PROCEEDINGS

OF THE

AMERICAN SOCIETY OF CIVIL ENGINEERS

VOL. 62 NOVEMBER, 1936 No. 9

TECHNICAL PAPERS

AND

DISCUSSIONS

Published monthly, except June and July, at 99–129 North Broadway, Albany, N. Y., by the American Society of Civil Engineers, Editorial and General Offices at 33 West Thirty-ninth Street, New York, N. Y. Reprints from this publication may be made on condition that the full title of Paper, name of Author, page reference, and date of publication by the Society, are given.

Entered as Second-Class Matter, December 28, 1931, at the Post Office at Albany, N. Y., under the Act of March 3, 1879. Acceptance for mailing at special rate of postage provided for in Section 1103, Act of October 3, 1917, authorized on July 5, 1918.

Subscription (if entered before January 1) $8.00 per annum. Price $1.00 per copy.

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AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

ECONOMIC DIAMETER OF STEEL PENSTOCKS

By Charles Voetsch[1], M. Am. Soc. C. E., and M. H. Fresen[2], Assoc. M. Am. Soc. C. E.

Synopsis

Formulas are developed in this paper for determining the economic diameter of steel penstocks for hydro-electric power plants and pumping plants based on a minimum total annual cost. The formulas developed differ from those published heretofore, in that:

(*a*) The Scobey formula for flow of water in riveted steel and analogous pipes is used;

(*b*) Factors are introduced to take care of entrance, bend, and other hydraulic losses which vary approximately as a ratio of the velocity head; and,

(*c*) A loss factor is introduced to take care of the effect of the load factor, and the shape of the load curve under which the power or pumping plant connected to the penstock operates.

The economic diameter of a penstock for a certain large power development as computed by formulas developed in this paper shows a difference of only 3.25% as compared to the diameter determined from detailed studies.

Definitions of Terms

Some writers consider the "pipe line" to be the pipe between the forebay and the surge tank, and the "penstock" to be the pipe between the surge tank and the turbines. Other writers use the term, "penstock", as applying to any pipe or pipes between intake or forebay at the dam and the turbines at the power-house of a water-power development, and "pipe line" as applying to a pipe used for other than water-power development, such as water supply.

The latter distinction between the terms, "pipe line" and "penstock", is implied herein, since the formulas are equally applicable to the pipe from forebay to surge tank, as from surge tank to turbines.

Note.—Discussion on this paper will be closed in March, 1937, *Proceedings*.

[1] Engr., U. S. Bureau of Reclamation, Denver, Colo. Mr. Voetsch died on February 7, 1935.

[2] Associate Engr., U. S. Bureau of Reclamation, Denver, Colo.

Notation.—The algebraic symbols used in this paper are defined where they are first introduced in the text, and are arranged for convenience of reference in the Appendix. Essentially, they conform to the American Tentative Standard Symbols for Hydraulics[3] compiled by a committee of the American Standards Association, with Society representation, and approved by the Association in 1929.

Annual Cost of Power Lost in Friction

In 1930, Fred C. Scobey, M. Am. Soc. C. E., introduced a formula[4] for head lost in friction, h_f, per 1 000 ft of steel pipe, as follows:

$$h_f = K_s \frac{V^{1.9}}{D^{1.1}} \qquad (1)$$

in which K_s is a general coefficient for head lost due to friction, in feet; V = the velocity of water in the penstock, in feet per second, corresponding to Q, the rate of discharge expressed in cubic feet per second; and D = the economic inside diameter of the penstock, in feet. Substituting for V its equivalent, $\frac{Q}{A} = \frac{Q}{0.7854\ D^2}$, and simplifying, Equation (1) becomes:

$$h_{f1} = \frac{K_s\ Q^{1.9}}{631.93\ D^{4.9}} \qquad (2)$$

in which h_{f1} = the head lost, in feet, due to friction in a penstock, 1 ft long; and A = the cross-sectional area.

The annual loss of power, in kilowatt-hours, at a load factor, F, per ft of pipe, is:

$$P_{f1} = Q\ h_{f1} \times \frac{62.5}{550} \times e \times 0.746 \times 8\,766 \times f \qquad (3)$$

in which e = the over-all efficiency of the plant to the point at which power is sold (in the case of a power plant), or to the point at which power is purchased (in the case of a pumping plant); and f = a loss factor, expressed as a decimal, corresponding to the load factor, F, under which a plant operates (average load ÷ maximum load). Substituting for h_{f1} its value as expressed by Equation (2), Equation (3) becomes:

$$P_{f1} = \frac{1.17631\ K_s\ Q^{2.9}\ e\,f}{D^{4.9}} \qquad (4)$$

The annual cost of the lost power, therefore, may be expressed as:

$$E_{f1} = P_{f1}\ b = \frac{1.17631\ K_s\ Q^{2.9}\ e\,f\,b}{D^{4.9}} \qquad (5)$$

in which E_{f1} = the annual cost of the lost power due to friction in a penstock, 1 ft long; and b = the value of the power lost by friction and other hydraulic factors expressed in dollars per kilowatt-hour.

[3] A. S. A.—Z 10 *b*—1929.

[4] "The Flow of Water in Riveted Steel and Analogous Pipes", by Fred C. Scobey, *Technical Bulletin No. 150*, U. S. Dept. of Agriculture, January, 1930, Equation (9), p. 10.

ANNUAL COST OF A PENSTOCK

If a penstock is operating under a head, H, an expression may be written for the thickness of the steel[5], t, as follows:

$$t = \frac{0.434\ H\ D \times 12}{2\ s_g\ e_j} \qquad \text{(6)}$$

in which s_g = the gross allowable tension in the steel penstock, in pounds per square inch; and e_j = the efficiency of the joints of the penstock, expressed as a decimal. The weight of a section, 1 ft long, W_1, in pounds, is:

$$W_1 = \frac{w\ \pi\ D\ t\ (1 + i)}{12} \qquad \text{(7)}$$

in which w = the weight of 1 cu ft of steel (= 490 lb); and i = the percentage by which the steel in the penstock is over-weight, due to laps, cover-plates, rivets, welds, etc., expressed as a decimal. Substituting Equation (6) in Equation (7) (with w = 490), and simplifying:

$$W_1 = \frac{334\ H\ D^2\ (1 + i)}{s_g\ e_j} \qquad \text{(8)}$$

The annual cost, E_{p1}, of a 1-ft length of penstock is:

$$E_{p1} = W_1\ a\ r = \frac{334\ H\ D^2\ a\ r\ (1 + i)}{s_g\ e_j} \qquad \text{(9)}$$

in which a = the unit cost of steel in the penstock, in dollars per pound; and r = the ratio of the annual fixed, operating and maintenance charges to the construction cost of the penstock.

ECONOMIC DIAMETER ANALYSES

Friction Loss and Annual Cost of Penstock Considered.—The annual cost, E_{t1}, of a 1-ft section of penstock is the sum of Equations (5) and (9). To determine the economic diameter, write out this sum and equate the first derivative with respect to D, to zero; thus,

$$\frac{d\ E_{t1}}{d\ D} = 1.17631\ K_s\ Q^{2.9}\ e\ f\ b\ (-4.9)\ D^{-5.9} + \frac{668\ H\ D\ a\ r\ (1 + i)}{s_g\ e_j} = -\frac{5.76392\ K_s\ Q^{2.9}\ e\ f\ b}{D^{5.9}} + \frac{668\ H\ D\ a\ r\ (1 + i)}{s_g\ e_j} = 0 \qquad \text{(10)}$$

Solving Equation (10) for D, and simplifying:

$$D = 0.50218\ \sqrt[6.9]{\frac{K_s\ Q^{2.9}\ e\ f\ b\ s_g\ e_j}{a\ H\ r\ (1 + i)}} \qquad \text{(11)}$$

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[5] "Water Power Engineering", by H. K. Barrows, M. Am. Soc. C. E., McGraw-Hill Book Co., 1927, p. 347.

Considering Losses Other Than Those Due to Friction.—Hydraulic losses, h_h, other than those due to friction, which vary approximately as a ratio of the velocity head corresponding to the velocity in the penstock, are defined by the relation:

$$h_h = K_T \frac{V^2}{2\,g} \qquad (12)$$

in which K_T = the sum of all constants for hydraulic losses, or,

$$K_T = k_b \sqrt{\frac{\Delta}{90}} + K_e + K_t \qquad (13)$$

in which K_e is a constant ruled by entrance losses; K_t is a constant to take care of all other hydraulic losses that vary as a ratio of the velocity head; k_b is a constant expressing the additional head lost due to the introduction of a bend in an otherwise straight penstock (usual approximate value = 0.25); $K_b = k_b \sqrt{\frac{\Delta}{90}}$; and, Δ = the angle of bend, in degrees.

When a bend is introduced in an otherwise straight penstock, the head lost, h_b, is expressed by:

$$h_b = k_b \sqrt{\frac{\Delta}{90}} \times \frac{V^2}{2\,g} = K_b \frac{V^2}{2\,g} \qquad (14)$$

If there is more than one bend in the pipe line under consideration, K_b will be the total of the values for all bends. The loss, h_e, at the entrance to a penstock is usually expressed as:

$$h_e = K_e \frac{V^2}{2\,g} \qquad (15)$$

Assuming a penstock of uniform diameter, with $V = \frac{Q}{A} = \frac{Q}{0.7854\,D^2}$, the head loss (Equation (12)) will be:

$$h_h = K_T \frac{V^2}{2\,g} = \frac{K_T\,Q^2}{2\,(0.7854)^2\,g\,D^4} \qquad (16)$$

Substituting Equation (16) in Equation (3), P_h for P_{f1}, and simplifying, the annual power loss, in kilowatt-hours, at the load factor, F, due to hydraulic factors, will be approximately:

$$P_h = \frac{18.70719\,K_T\,Q^3\,e\,f}{D^4} \qquad (17)$$

and, consequently, the annual cost of power loss, E_h, due to hydraulic factors other than friction (compare Equation (5)) will be:

$$E_h = P_h\,b = \frac{18.70719\,K_T\,Q^3\,e\,f\,b}{D^4} \qquad (18)$$

Considering Friction Loss, Other Hydraulic Losses, and Annual Cost of Penstock.—For this case it will be necessary to determine all hydraulic losses,

including those due to friction, and the annual cost, E_p, of the entire penstock of length, L, in feet. The annual cost of the friction loss per foot of penstock is expressed by Equation (5). Let the length of penstock used for computing friction loss be L_f, and the annual cost of friction loss in the entire line will be:

$$E_f = E_{f1}\, L_f = \frac{1.17631\, K_s\, Q^{2.9}\, e\, f\, b\, L_f}{D^{4.9}} \qquad (19)$$

Likewise, the annual cost of steel per foot of penstock is expressed by Equation (9). Let the length of penstock used for computing total weight be L_p, and the annual cost of the steel in the line will be:

$$E_p = E_{p1}\, L_p = \frac{334\, H\, D^2\, a\, r\, L_p\, (1 + i)}{s_g\, e_j} \qquad (20)$$

The total annual cost, E_t, of the entire penstock will be the sum of Equations (18), (19), and (20), or:

$$E_t = E_f + E_h + E_p \qquad (21)$$

To determine the economic diameter equate the derivative of the total annual cost, E_t, with respect to D, to zero; thus:

$$\frac{d\,E_t}{d\,D} = -\frac{5.76392\, K_s\, Q^{2.9}\, e\, f\, b\, L_f}{D^{5.9}} - \frac{74.82876\, K_T\, Q^3\, e\, f\, b}{D^5} + \frac{668\, H\, D\, a\, r\, L_p\, (1 + i)}{s_g\, e_j} = 0 \qquad (22)$$

In Equation (22), let,

$$G = 5.76392\; K_s\; Q^{2.9}\; L_f \qquad (23a)$$

$$M = 74.82876\; K_T\; Q^3 \qquad (23b)$$

$$J = e\; f\; b \qquad (23c)$$

and,

$$B = \frac{668\, H\, a\, r\, L_p\, (1 + i)}{s_g\, e_j} \qquad (23d)$$

Then, substituting and solving for B, Equation (22) becomes of the form:

$$B = \frac{G\,J}{D^x} + \frac{M\,J}{D^y} \qquad (24)$$

in which x and y are different exponents of D. Because of this difference, Equation (24) cannot be solved directly; but since the difference between them is only 13%, it is possible to assume that $x = y$, so that $B = \frac{J\,(G + M)}{D^x}$;

or, approximately:

$$D = \sqrt[x]{\frac{J\,(G + M)}{B}} \quad \ldots\ldots(25)$$

Substituting $x = 6.9$, Equation (25) can be written as:

$$D = \sqrt[6.9]{\frac{J\,(G + M)}{B}} \quad \ldots\ldots(26)$$

approximately, or, in terms of all the fundamental factors involved:

$$D = \sqrt[6.9]{\frac{e\,f\,b\,(5.76392\,K_s\,Q^{2.9}\,L_f + 74.82876\,K_T\,Q^3)\,s_g\,e_j}{668\,H\,a\,r\,L_p\,(1 + i)}} \quad \ldots\ldots(27)$$

and, for greater accuracy, Equation (24) (with $x = 6.9$ and $y = 6.0$) can be solved for D by a trial-and-error method.

Considering Thickness of Shell.—In some instances, the thickness of the shell or of the plates of a penstock will be given instead of the head. As before (see Equation (9)) the annual cost of a 1-ft length of penstock is $E_{p1} = W_1\,a\,r$. Substituting the value of W_1 from Equation (7) (with $w = 490$), and simplifying:

$$E_{p1} = 128.282\,t\,D\,a\,r\,(1 + i) \quad \ldots\ldots(28)$$

The annual cost of a penstock of length, L_p, is:

$$E_p = E_{p1}\,L_p \quad \ldots\ldots(29)$$

Substituting Equation (28) in Equation (29) and finding the derivative of E_p with respect to D:

$$\frac{d\,E_p}{d\,D} = 128.282\,t\,a\,r\,L_p\,(1 + i) = B' \quad \ldots\ldots(30)$$

in which B' is a substitution factor. In Equation (22), substitute Equation (30) for the term, $\frac{668\,H\,D\,a\,r\,L_p\,(1 + i)}{s_g\,e_j}$, so that B in Equation (24) becomes $\frac{B'}{D}$, and,

$$B' = \frac{G\,J}{D^{x-1}} + \frac{M\,J}{D^{y-1}} \quad \ldots\ldots(31)$$

If $x = y = 6.9$ as before, and if the fundamental factors for J, G, M, and B' are substituted logically, Equation (31) may be solved for D in a manner similar to Equation (26):

$$D = \sqrt[5.9]{\frac{J\,(G + M)}{B'}} \quad \ldots\ldots(32)$$

approximately, or, in terms of all the fundamental factors involved:

$$D = \sqrt[5.9]{\frac{e\,f\,b\,(5.76392\,K_s\,Q^{2.9}\,L_f + 74.82876\,K_T\,Q^3)}{128.282\,t\,a\,r\,L_p\,(1 + i)}} \quad \ldots\ldots(33)$$

In order to apply Equation (32) to a practical design, compute the constants, J, G. and M, and determine the economical diameter for different values of t in B' (see Equation (30)). With t and D known the permissible head may be computed from Equation (6) and this value determines the economic diameter at each station on the line where the head corresponds to the values of t and D. As in the case of Equation (24), Equation (31) (with ($x = 6.9$ and $y = 6.0$) can be solved for D with greater accuracy by trial and error.

Accuracy of Equations (26) and (32)

Computations show that the errors in the values of D computed from Equations (26) and (32), for $M = 0.5\ G$ and $M = G$, will not exceed -0.528% and -0.973%, respectively, compared to the accurate solution of those equations. For $M = 0$ corresponding to $K_T = 0$ and with $L = 1$ ft, the equations as given are accurate.

Relative Values of Economic Diameter

Table 1 illustrates the effect of varying the values of the expressions under the radical signs of Equations (11), (26), and (32). It shows that a considerable change in the individual values of the constants in the equations is necessary in order to affect the diameter appreciably. A change in the discharge is much more noticeable. Hence, extreme accuracy is not essential in any particular study, except that the discharge should be estimated as closely as possible.

TABLE 1.—Relative Values of Economic Diameter

Relative value of entire expression under radical sign, Equations (11), (26), or (32)	Relative Economic Diameter, D		Relative value of entire expression under radical sign, Equations (11), (26), or (32)	Relative Economic Diameter, D	
	Computed by Equations (11) and (26) $x = 6.9$	Computed by Equation (32) $x = 5.9$		Computed by Equations (11) and (26) $x = 6.9$	Computed by Equation (32) $x = 5.9$
(1)	(2)	(3)	(1)	(2)	(3)
0	0	0	1.25	1.033	1.039
0.25	0.818	0.791	1.50	1.061	1.071
0.50	0.904	0.889	1.75	1.085	1.100
0.75	0.959	0.952	2.00	1.106	1.125
1.00	1.000	1.000	5.00	1.263	1.314

Comparison of Results

In 1934, detailed studies were made of a large power development to determine the economic diameter of the penstocks. Subsequently, formulas were developed as outlined herein, using the same general principles as in the detailed studies. For convenience of comparison, the methods used in determining the economic diameter for this development are designated as A, B, and C. The same basic data were used in all three methods.

In Method A the annual cost of friction losses and of the penstock steel, were considered. The annual cost of the entire length of penstock, and bend and entrance losses, in addition to the friction losses, were considered in

Method B; and, in Method C, a detailed study was made of the economic diameter of the penstock. Included in the latter were the annual costs of the penstock, friction, bend, and entrance losses as in Method B and, in addition, annual costs due to rack, valve, and cone losses. Water-hammer heads were computed separately for each assumed size of penstock. The economic size of the penstock, based on the minimum annual cost read from the Diameter-Total Annual Cost Curve, was 19.67 ft. The results of the computations by the three methods are summarized in Table 2.

TABLE 2.—COMPARISON OF METHODS TO DETERMINE ECONOMIC DIAMETER OF PENSTOCK

Method (1)	Equation used (2)	Diameter, D, in feet (3)	Percentage Error Compared with Method C: Diameter (4)	Percentage Error Compared with Method C: Total annual cost (5)
A	(11)	18.29	−7.02	+2.22
B	(26)	19.03	−3.25	+0.65
C	Detailed study	19.67	0	0

For this particular development, the bend losses formed a large proportion of the total losses. Hence, since bend losses are neglected entirely in Method A, it can be readily seen that the value of D, computed by Method B, should be larger than when computed by Method A. In the same way, Method C includes other losses in addition to those of Method B, and hence D, by Method C, is larger than by Method B. Another factor that must be considered is that Equation (26), for Method B, is approximate, and a more accurate solution would give a slightly higher value for D. Column (5), Table 2, shows the percentage errors in total annual cost of Methods A and B compared to Method C. The values of total annual cost, used to determine the percentage errors, were read from the Diameter-Total Annual Cost Curve for diameters corresponding to the three methods.

Table 2 shows that consideration of additional hydraulic losses and the resulting cost of lost power justifies additional capital expenditure for a larger penstock.

Use of Formulas.—The data for the illustrative example are, as follows: $Q = 400$ cu ft per sec; $e = 0.75$; $K_s = 0.401$; $K_e = 0.05$; $k_b = 0.25$; $\Delta = 25.9°$; $K_T = K_e + K_b = 0.05 + 0.1342 = 0.1842$; $L_p = 100$ ft; $L_f = 130$ ft; $H = 130$ ft; $a = \$0.10$ per lb; $r = 0.17$; $s_g = 12\,000$ lb per sq in.; $e_j = 0.90$; $i = 0.20$; $f = 0.25$; and $b = \$0.007$ per kw-hr.

The entire length of the penstock, and the bend and entrance losses, in addition to the friction losses, will be considered in the example. Equation (26) is used to determine the economic diameter, D; but, first, it is necessary to solve for the several constants.

Substituting the foregoing values in Equations (23), and simplifying, $J = 13.14 \times 10^{-4}$; $G = 1\,058 \times 10^{7}$; $M = 88.3 \times 10^{7}$; and, $B = 16.4$. Then, from Equation (26), $D = 7.315$ ft; say, 7 ft 4 in.

Choice of Formula for Friction Loss

The Scobey formula (Equation (1)) was selected because it was the most authoritative and the newest available. Mr. Scobey deduced it after an extensive series of tests and a complete review and analysis of previous investigations. Among other factors, he allowed for the kinematic viscosity of water at 15° C (59° F), and for Reynolds' number. He also recommended proper coefficients for use in his formula, based on various classes of pipes and joints and ages of pipe, the coefficients being constant throughout the usual ranges of pipe diameters and water velocities.

Other Formulas for Economic Diameter of Steel Penstocks

Other formulas have been introduced by M. L. Enger[6], and H. K. Barrows[7], Members, Am. Soc. C. E., G. E. Lyon[8], Assoc. M. Am. Soc. C. E., and by Messrs. R. L. Daugherty[9], R. M. Peabody[9], B. F. Jakobsen[10], and W. F. Durand[11].

The Enger formula, as modified somewhat by W. P. Creager and J. D. Justin[12], Members, Am. Soc. C. E., is quite similar to Equation (33) except that they used the Chezy formula for friction loss. Professor Barrows gives a formula similar to Equation (11) except that he used the Weisbach formula for friction and assumed a value of efficiency, e, equal to 1.0. H. L. Doolittle, M. Am. Soc. C. E., outlines a method for the economic design of steel penstocks[13] and in their discussion[9] of that paper, Messrs. Daugherty and Peabody develop analytical solutions for the economic diameter.

Of the foregoing writers, none includes hydraulic losses other than those due to friction, and only Messrs. Creager and Justin attempt to give any specific data on the variation in yearly losses with the annual load factor.

Economic Velocity

Table 3 summarizes computations made to determine the most economic maximum velocity, V, for maximum penstock discharges, Q, of 400, 2 000 and 5 000 cu ft per sec, and heads for shell thickness, H, of 100 and 400 ft for an assumed set of conditions. The assumptions made for the computations are as follows: $K_s = 0.400$; $e = 0.85$; $f = 0.20$ corresponding approximately to a load factor, F, of about 50%; $b = \$0.003$ per kw-hr; $s_g = 15\,000$ lb per sq in.; $e_j = 0.80$; $a = \$0.10$ per lb; $r = 0.07$; $i = 0.10$; and t = minimum shell thickness = $\frac{5}{16}$ in. A 1-ft length of penstock was assumed. Frictional losses

[6] "Economic Design of Penstocks", by M. L. Enger, *Engineering Record*, September 12, 1914, p. 300; also, Editorial in same issue indicating application to concrete pipes; also, inquiry by Mr. C. R. Steiner about article in *Engineering Record*, October 31, 1914, p. 495, and reply by Dean Enger in the same issue, pp. 495–496.

[7] "Water Power Engineering", by H. K. Barrows, 1927, p. 344.

[8] "Economic Size of Pipe for Power Purposes", by G. E. Lyon, *Engineering and Contracting*, May 10, 1922, pp. 436–437.

[9] *Transactions*, A. S. M. E., Vol. 46, 1924 (Paper No. 1946), pp. 1178–1203.

[10] "Economic Pressure Pipe on Penstock Pipe Design", by B. F. Jakobsen, *Engineering and Contracting*, December 11, 1918, pp. 554–556.

[11] "Hydraulics of Pipe Lines", by W. F. Durand, D. Van Nostrand Co., 1921.

[12] Hydro-Electric Handbook, by W. P. Creager and J. D. Justin, John Wiley & Sons, 1927, p. 414.

[13] "A Method for the Economic Design of Penstocks", by H. L. Doolittle, *Transactions*, A. S. M. E., Vol. 46, 1924 (Paper No. 1946), pp. 1165–1178.

only were considered. Except for Item No. 1, Table 3, required shell thicknesses as computed from Equation (6) were in excess of the $\frac{5}{16}$-in. minimum.

TABLE 3.—SUMMARY OF COMPUTATIONS FOR ECONOMIC VELOCITY

Item No. (1)	Maximum discharge, Q, in cubic feet per second (2)	Equation used to compute D (3)	Minimum shell thickness, t, in inches (4)	Economic diameter, D, in feet (5)	Economic velocity, V, in feet per second (6)	Item No. (1)	Maximum discharge, Q, in cubic feet per second (2)	Equation used to compute D (3)	Minimum shell thickness, t, in inches (4)	Economic diameter, D, in feet (5)	Economic velocity, V, in feet per second (6)
H = 100 FEET FOR SHELL THICKNESS						H = 400 FEET FOR SHELL THICKNESS					
1	400	(33)	$\frac{5}{16}$	7.40	9.31	4	400	(11)	Not required	6.03	14.03
2	2 000	(11)	Not required	14.49	12.13	5	2 000	(11)	Not required	11.85	18.13
3	5 000	(11)	Not required	21.30	14.04	6	5 000	(11)	Not required	17.42	20.98

Table 3 shows that V increases with Q and H. Inspection of the velocities in Column (6) indicate that they are fairly representative of existing practice. Economic velocities for conditions other than those assumed may be found in a similar manner, or proportioned from the results given in the table (see Equations (11) and (33)).

ANNUAL POWER LOSS, LOAD FACTOR, AND LOSS FACTOR

The average annual power loss is designated herein as P_a, and the maximum annual power loss for load factor, $F = 1.0$, is designated as P_m. Corresponding discharges are Q_a and $Q_m = Q$. Since discharge is assumed as proportional to power generated,

$$F = \frac{Q_a}{Q_m} = \frac{Q_a}{Q} \qquad \text{(34)}$$

For a penstock of a given diameter, the head lost varies as $V^{1.9}$ and $Q^{1.9}$. The power lost varies as the product of head lost times discharge. Hence, the power lost varies as $Q^{2.9}$ for friction; and as Q^3 for bend, entrance, and other hydraulic losses where h_h varies as V^2 and Q^2. The equivalent annual discharge which will give a power loss, P_a, corresponding to the load factor, F, and average discharge, Q_a, is:

$$Q_e = \sqrt[2.9]{\frac{q^{2.9}{}_1 + q^{2.9}{}_2 + q^{2.9}{}_3 + \ldots q^{2.9}{}_n}{N_1 + N_2 + N_3 + \ldots N_n}} \qquad \text{(35)}$$

in which q_1, q_2, etc., are the discharges of the penstock, in cubic feet per second, for N_1, N_2, etc., hr, for the given period and load curve. For simplicity the exponent and root in Equations (35) and (36) may also be taken as 3 instead of 2.9 if desired. The relations between power losses, discharges, and loss factor is expressed by:

$$f = \frac{P_a}{P_m} = \frac{j\,Q^{2.9}_e}{j\,Q^{2.9}} \qquad \text{(36)}$$

in which j is a constant. For a 1-ft length of penstock, a given diameter, D, and $f = 1.0$ for $F = 1.0$, the value of the constant, j, in Equation (36) is

determined from Equation (4) to be:

$$j = 1.17631\ K_s\, e \qquad (37)$$

Messrs. Creager and Justin present data[14] and formulas which, reduced to the symbols of this paper, are as follows:

$$Q^3_a\, C_r = Q^3_e = Q^3_f \qquad (38)$$

Their coefficient, C_r, varies with the load factor, F, and is given in their Fig. 43, which is used in their formula for the economic diameter of a steel

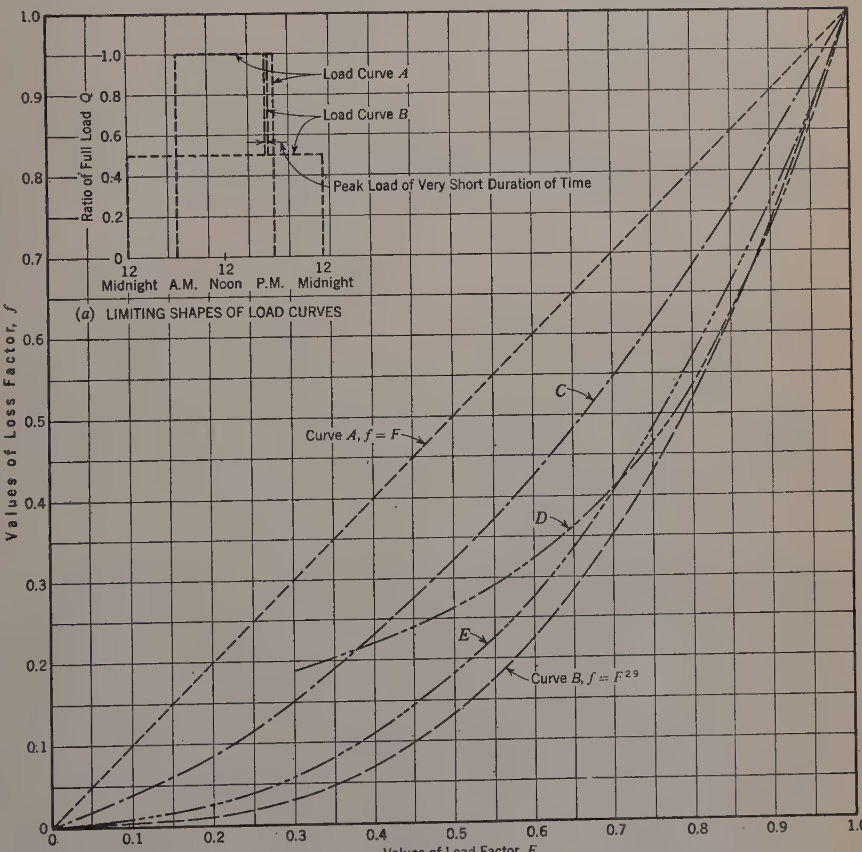

FIG. 1.—LOAD FACTOR-LOSS FACTOR CURVES FOR WATER CONDUITS

penstock.[15] Assuming that $F = \dfrac{Q_a}{Q}$, a loss factor was computed and plotted as Curve D, in Fig. 1, from the relation:

$$f = F^3\, C_r \qquad (39)$$

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[14] Hydro-Electric Handbook, pp. 82 and 83, Equation 19, and Fig. 43.

[15] *Loc. cit.*, p. 414.

Loss Factor Data and Curves

Fig. 1 establishes the limiting loss factor curves, A and B, for steel penstocks corresponding to the given limiting shapes of Load Curves A and B in Fig. 1(a). Load Curve A is of rectangular shape and of constant peak load, whereas, Load Curve B is of rectangular shape with a peak load of uniform and very short duration, extending above the rectangular part. The equations of Curves A and B are, respectively, $f = F$ and $f = F^{2.9}$. Messrs F. H. Buller and C. A. Woodrow established similar limiting loss factor curves for transmission lines.[16] Curve C is taken from their average experience loss factor curve for transmission lines, the basic plotted points for which were computed from a series of actual and representative load curves of public utility plants serving diversified loads.

Curve D is an attempt to establish an average loss factor curve from data given by Messrs. Creager and Justin.[14] Curve E is a representative loss factor curve for water conduits, such as penstocks and tunnels, carrying water for public utility hydro-electric power plants serving typical diversified loads. It has been computed as follows: For any given load factor, F:

$$f_E = f^{1.5}c \qquad (40)$$

in which f_C = the loss factor for transmission lines from Curve C for a given load factor, and f_E = the loss factor for water conduits. For cases in which a fairly accurate load curve can be established, the longer method of determining a value of Q_e should be used (see Equation (35)).

In Fig. 1, the value of Q for full-load water use is assumed to be that for full gate-opening, or for the rated horse-power of a turbine under its rated head. The friction head varies as $V^{1.9}$ or $Q^{1.9}$, and the power loss varies as $Q^{2.9}$. The power generated is assumed to be proportional to the water use. Until further data are available, the value of f for a typical load curve can be assumed to lie between Curves C and E, probably nearer to Curve E.

The curves of Fig. 1 may be used to estimate values of f for all water conduits in which the friction head varies approximately as V^2. The load factor, F, may be on a daily, weekly, monthly, or yearly basis. Mr. J. G. Tarboux uses a loss factor in formulas he develops for the economic size of transmission line conductors[17], in a manner similar to the method described in this paper, for the economic diameter of a steel penstock.

Table 4 summarizes computations made to check the accuracy of Equation (40) and of Curve E, Fig. 1, as applied to water conduits. A typical load curve[18] of the Alabama Power Company's system (mainly hydro-electric generation) was used as a basis. Its load factor was 0.795. Load curves for load factors of 0.321 and 0.550 were also drawn with shapes similar to that

[16] "Load Factor—Equivalent Hour Values Compared", by F. H. Buller and C. A. Woodrow, *Electrical World*, July 14, 1928, p. 59; also, *Electrical World*, August 18, 1928, p. 320.

[17] "Most Economical Conductor", by J. G. Tarboux, *Electrical World*, March 23, 1929.

[18] "Determining Transmission Line Losses" by S. Murray Jones and Joel Tompkins, *Electrical World*, November 17, 1928, p. 993.

for a load factor of 0.795, except for the peak load. The loss factors for transmission lines in Column (3), Table 4, were computed on the assumption that the power loss varies as the square of the hourly ordinates (or load) of the load curve. The loss factors for water conduits in Column (5) were computed on the assumption that the power loss varies as the cube of the hourly ordinates (or load). Inspection of Table 4 shows that Equation (40) and Curve *E*, Fig. 1, are applicable in all ordinary cases.

TABLE 4.—COMPARISON OF LOSS FACTORS FOR LOAD CURVES

Item No.	Load factor, *F*	Loss Factor *f*			
		Transmission Lines		Water Conduits	
		Computed from load curve	Read from Fig. 1, Curve *C*	Computed from load curve	Read from Fig. 1, Curve *E*
(1)	(2)	(3)	(4)	(5)	(6)
1	0.321	0.144	0.166	0.082	0.067
2	0.550	0.337	0.373	0.225	0.228
3	0.795	0.655	0.677	0.561	0.563

ECONOMIC DIAMETER OF PENSTOCK FROM TOTAL ANNUAL COST CURVE

When it is desired to include costs other than those of, and in addition to, the frictional and hydraulic losses, and penstock steel, as given in the formulas heretofore developed, the procedure will be to assume several different penstock diameters and determine the annual cost for each diameter, including all the factors and elements of cost that vary with the diameter. These additional costs which are affected by the penstock diameter include: (*a*) Intake cost; (*b*) penstock valve costs; (*c*) annual saving in cost of concrete in the dam by using a larger penstock diameter; (*d*) cost of required generator flywheel effect for machine of desired rating, if more than that usually provided in a generator of normal design; (*e*) cost of surge tank, if any; (*f*) governor cost; and (*g*) cost of penstocks supports, etc. Referring to Item (*d*), the flywheel effect, is usually called $W R^2$, which equals the product of the rotating weight of generator rotor, in pounds, and the square of the radius of gyration, in feet.

The totals of the annual costs for all items for each assumed diameter are plotted as ordinates, with penstock diameters as abscissas. The minimum annual cost is found from the Diameter-Total Annual Cost Curve. For a study of this type, Equations (18), (19) and (20) will be found convenient to determine the annual costs of the frictional and hydraulic losses, and penstock steel, respectively.

ECONOMIC DIAMETER OF PENSTOCKS FOR PUMPING PLANTS

In general, the formulas developed in this paper for the economic diameter of steel penstocks for hydro-electric power plants apply to electrically operated pumping plants. In such a case, the annual cost of the power consumed in friction and other hydraulic losses, varies inversely as the over-all efficiency

of the plant, e, since for a lower efficiency more power must be used to pump a given quantity of water against a given total head. On the other hand for a power plant, the annual cost of the power lost in friction and other hydraulic losses varies directly as the over-all efficiency, e, as shown by Equation (5). In Equation (11) for the economic diameter, e appears in the numerator of the expression under the radical sign, and in Equations (26) and (32), e as one factor in J appears in the numerator of the expression under the radical sign.

It follows, therefore, that in using the foregoing formulas to determine the economic diameter of penstock for a pumping plant, the efficiency, e, should be placed in the denominator of Equations (11), (26), and (32). In these formulas it is assumed that the head consumed in friction and other hydraulic losses forms only a small part of the static pumping head, and, therefore, that the size of the pumping motors and plant remains constant for a reasonable variation in the size of the penstock.

For a pumping plant with a very long penstock, and relatively low static head, it would be better to use the method outlined under the heading "Economic Diameter of Penstock from Total Annual Cost Curve", and include the annual cost of the pumping plant itself, and other variables, for each assumed diameter or plate thickness.

Conclusions

The formulas developed in this paper are general, but they are intended to apply particularly to the determination of the economic size of a penstock of uniform diameter for a hydro-electric power plant, particularly where the power-house is situated immediately down stream from the penstock intake section of the dam. Equation (11) or Equation (26) is to be used, depending on the accuracy desired, or the amount of data available.

For a long penstock, the diameter generally decreases toward the lower end, and the penstock is usually made in sections of uniform diameter and the same plate thickness, with transitions connecting the sections of different diameters. Equation (32) applies to this case.

Acknowledgments

The subject of, and formulas developed in, this paper were suggested by studies made by the writers for a large power development. The senior writer, the late Mr. Charles Voetsch, passed away before he could review the paper entirely. The junior writer wishes to express herein his appreciation of the fine and unfailing spirit of co-operation and friendliness with which Mr. Voetsch drew on his wealth of experience and his valuable engineering library to assist or furnish data to all who desired aid in special problems. I. A. Winter, Assoc. M. Am. Soc. C. E., Senior Engineer, U. S. Bureau of Reclamation, made several valuable suggestions which have been incorporated in the paper. Power plant studies and designs of the U. S. Bureau of Reclamation, including those upon which this paper is based, are under the direction of L. N. McClellan, Chief Electrical Engineer. All studies and designs are under the general direction of J. L. Savage, M. Am.

Soc. C. E., Chief Designing Engineer; all engineering and construction work is under the direction of R. F. Walter, M. Am. Soc. C. E., Chief Engineer, with headquarters at Denver, Colo.; and all activities of the Bureau at the time this paper was prepared were under the general charge of the late Elwood Mead, M. Am. Soc. C. E., Commissioner of Reclamation. John C. Page, Acting Commissioner of Reclamation, with headquarters in Washington, D. C., is in general charge of Bureau activities.

APPENDIX

Notation

The symbols introduced in the paper are defined as follows:

a = the cost of steel in the penstock, in dollars per pound.
A = the cross-sectional area in a penstock, in square feet.
b = the value of power loss because of friction and other hydraulic factors, in dollars per kilowatt-hour.
B = a constant in Equation (26), defined by Equation (23*d*); B' = a constant in Equation (32), defined by Equation (30).
C_r = the coefficient used by Messrs. Creager and Justin, defined by Equation (38).
D = the economic inside diameter of a penstock, in feet.
e = the over-all efficiency of a plant to: (1) The point where power is sold, for a power plant; and (2) the point where power is purchased, for a pumping plant; e_j = the efficiency of the joints of a penstock, expressed as a decimal.
E = the annual cost (expense) of a penstock of length, L, in dollars; E_f = the annual cost of the power lost because of friction, in a penstock of length, L_f; E_{f1} = the annual cost of the power lost because of friction in a penstock, 1 ft long; E_h = the annual cost of the power lost because of hydraulic factors other than friction (such as entrance losses, bend losses, etc.); E_p = the annual cost of the steel in a penstock of length, L_p; E_{p1} = the annual cost of the steel in a penstock, 1 ft long; E_t = the total annual cost of a penstock of length, L; E_{t1} = the total annual cost of a penstock, 1 ft long.
f = a loss factor, expressed as a decimal, corresponding to the load factor, F, and defined by Equations (36) and (38).
F = the load factor under which the plant operates (= average load ÷ maximum load); for a discharge proportional to the power generated, $F = Q_a \div Q$.
g = the acceleration due to gravity (= 32.16).
G = an algebraic substitution factor in Equations (26) and (32), as defined by Equation (23*a*).
h = loss of head, in feet; h_f = the head lost in friction, in a penstock of length, L_f; h_{f1} = the head lost in friction, in a penstock 1 ft long; h_h = the head lost because of hydraulic factors other than friction; h_t = the total head lost because of friction and hydraulic factors; h_b = the additional head lost, due to the introduction of a bend in an otherwise straight penstock; h_e = the head lost at the entrance to a penstock or intake.

H = the average head, on the penstock, including water-hammer effect, in feet (weighted average to be used if necessary).

i = percentage of overweight of steel in a penstock, due to laps, cover-plates, rivets, welds, etc., expressed as a decimal.

j = a constant in Equation (36) defined by Equation (37).

J = an algebraic substitution factor in Equations (26) and (32), as defined by Equation (23c).

k_b = a constant in Equation (14) defined by $k_b = K_b \div \sqrt{\frac{\Delta}{90}}$; usual approximate value = 0.25.

K = a coefficient; K_b = a constant that modifies the expression for head loss when bends are introduced in an otherwise straight pipe (see Equation (14)); K_e = a constant that modifies the expression for head loss at the entrance to a penstock (see Equation (15)); K_s = a general coefficient in the Scobey formula (Equation (1)) which determines the head lost in friction; K_t = a constant in Equation (13) to account for hydraulic effects other than bend losses and entrance losses; K_T = the sum of all constants for hydraulic losses, as defined by Equation (13).

L = length, in feet: L_f = length of penstock used to compute friction loss when it differs from L_p; L_p = length of penstock used to compute the weight of steel when it differs from L_f;

M = an algebraic substitution factor in Equations (26) and (32), as defined by Equation (23b).

N = time, in hours, for a given period and load curve, during which the penstock discharge is q (q_1, q_2, etc., correspond with N_1, N_2, etc.).

P = annual power lost, in kilowatt-hours: P_a = average loss due to friction and hydraulic factors, at the load factor, F; P_f = loss due to friction in a penstock of length, L_f; P_{f1} = loss due to friction in a penstock 1 ft long; P_h = loss due to hydraulic factors other than friction; P_m = loss due to friction and other hydraulic factors, at full or maximum load, corresponding to the discharge, Q, and the load factor, $F = 1.0$.

q = discharge of a penstock, in cubic feet per second, during a stated time, N, and for a given period and load curve; q_1, q_2, etc., correspond to N_1, N_2, etc.

Q = the rated discharge of a penstock, in cubic feet per second, corresponding to full gate-opening, or the rated horse-power of the connected turbine or turbines (or as limited by the generator capacity), when operating under rated or normal head (in this paper, penstock discharge is assumed to be proportional to the power generated, and Q is assumed equal to Q_m); Q_a = average discharge corresponding to the average power generated during the period and for the load curve considered; Q_e = the equivalent discharge, for the period and load curve considered, to give a power loss for the period equivalent to the actual power loss for the period and the load curve considered (see Equation (35)); Q_m = the maximum discharge for the load factor, $F = 1.0$ (Q_m is assumed equal to Q).

r = the ratio of the annual fixed, operating, and maintenance charges to the construction cost of the penstock.

s = unit stress, in pounds per square inch; s_g = the gross allowable tension in the penstock steel.
t = the thickness of the steel in the penstock, in inches.
V = the velocity of the water in the penstock, in feet per second, corresponding to the discharge, Q.
w = weight of steel, in pounds per cubic foot (= 490).
W = weight of steel, in pounds, in a section of a penstock of length, L; W_1 = weight of steel in a 1-ft section of a penstock.
x = a variable exponent of diameter, D (see Equations (24), (25), and (31)).
y = a variable exponent of diameter, D (see Equations (24) and (31)).
Δ = the angle of bend in a penstock, in degrees.

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PAPERS

STRESSES AROUND CIRCULAR HOLES IN DAMS AND BUTTRESSES

By I. K. Silverman[1], Jun. Am. Soc. C. E.

Synopsis

Cracks that have been observed in dams have sometimes been attributed to the presence of drainage or inspection galleries which are parallel or transverse to the axis of the dam. Some observers have stated that these cracks generally start from the gallery and in some cases travel to both down-stream and up-stream faces, thus dividing the structure into two or more parts. In other cases these cracks are purely local and extend only a short distance from the gallery into the main part of the dam. Both these types of cracks can be observed on the faces of buttresses.

The stresses which occur around these openings are due to volumetric changes arising from shrinkage, temperature, etc., and to water and mass loads. An attempt is made in this paper to analyze the stresses around a circular opening caused by water and mass loads.

Notation.—In the notation introduced herein (see Appendix I) an effort has been made to conform to the American Standard Symbols for Mechanics, Structural Engineering, and Testing Materials[2], compiled by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1932.

General Formulas

Both theory and experiment[3] have shown that, in a large body, the effect of a relatively small opening is a purely local one. The stresses at points that are removed a sufficient distance from the discontinuity are essentially the same as if there were no hole. A case that has been investigated to some

Note.—Discussion on this paper will be closed in **March, 1937,** *Proceedings.*

[1] With U. S. Bureau of Reclamation, Denver, Colo.

[2] A. S. A.—*Z* 10*a*—1932.

[3] "Photo-Elasticity", by E. G. Coker and L. N. G. Filon, pp. 481 *et seq.*, Cambridge Press, 1931.

extent is that of a circular hole in a bar subjected to uniform tension. The normal stress, s_{nx}, on Section AA of Fig. 1[4] is given by,

$$s_{nx} = \frac{s}{2}\left(2 + \frac{r^2_h}{\rho^2} + 3\,\frac{r^4_h}{\rho^4}\right) \quad \text{.................(1)}$$

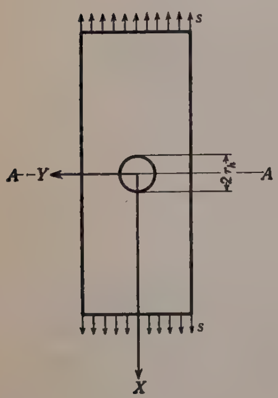

FIG. 1

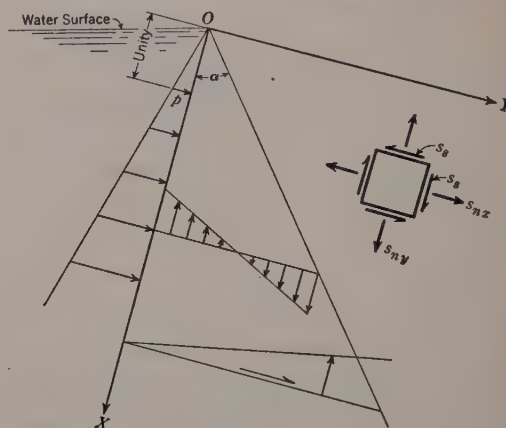

FIG. 2

in which s = the total load divided by the gross area of the bar; r_h = the radius of the hole in the bar; and ρ = radial distance from center of hole. As ρ increases s_{nx} approaches s. In the immediate vicinity of the hole, however, the stresses are larger than those calculated as if no hole existed; and $s_{nx} = 3\,s$ in Equation (1) if $\rho = r_h$.

STRESSES DUE TO WATER LOAD

Although openings in dams or buttresses are not always circular the state of stress around such openings may be examined on the assumption that they are circular. In the analysis which follows the usual assumptions of homogeneity, elastic behavior, and the applicability of Hooke's law, are made. Triangular dams and buttresses subjected to hydrostatic loadings are usually designed on the assumption that the normal stresses, s_n, on a section are distributed linearly (see Fig. 2) and are given by,

$$s_n = \frac{p\,x}{K^2} - 2\,\frac{p\,y}{K^3} \quad \text{........................(2)}$$

in which p = pressure at a unit distance measured from O along the up-stream face (Fig. 2); K = a substitution factor = tan α; α = the angle between the battered faces of a dam; and x and y are variable distances measured in the X- and Y-directions, respectively. The shear stresses, s_s (see Fig. 2), are given by,

$$s_s = -\,\frac{p\,y}{K^2} \quad \text{.........................(3)}$$

[4] "Theory of Elasticity", by S. Timoshenko, p. 78.

Considered as a two-dimensional problem, the equations of equilibrium of a rectangular particle are:

$$\frac{\partial s_x}{\partial x} + \frac{\partial s_s}{\partial y} = 0 \quad \text{.......................(4)}$$

and,

$$\frac{\partial s_y}{\partial y} + \frac{\partial s_s}{\partial x} = 0 \quad \text{.......................(5)}$$

To preserve geometrical continuity under Hooke's law and to satisfy the equations of equilibrium, the stresses, s_x, s_y, and s_s, are defined as follows:

$$s_x = \frac{\partial^2 F}{\partial y^2} \quad \text{.......................(6a)}$$

$$s_y = \frac{\partial^2 F}{\partial x^2} \quad \text{.......................(6b)}$$

and,

$$s_s = -\frac{\partial^2 F}{\partial x \; \partial y} \quad \text{.......................(6c)}$$

in which, F, the Airy stress function, is given by the following:

$$\frac{\partial^4 F}{\partial x^4} + 2\,\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \quad \text{.................(7)}$$

A solution of Equation (7) that satisfies all boundary conditions for a triangular dam section loaded as shown in Fig. 2 is:

$$F = \frac{p\,x\,y^2}{2\,K^2} - \frac{p\,y^3}{3\,K^3} - \frac{p\,x^3}{6} \quad \text{.................(8)}$$

from which the stresses by Equation (1) are found to be:

$$s_x = \frac{\partial^2 F}{\partial y^2} = \frac{p\,x}{K^2} - \frac{2\,p\,y}{K^3} \quad \text{.................(9a)}$$

$$s_y = \frac{\partial^2 F}{\partial x^2} = -\,p\,x \quad \text{.......................(9b)}$$

and,

$$s_s = -\frac{\partial^2 F}{\partial x \; \partial y} = -\frac{p\,y}{K^2} \quad \text{.................(9c)}$$

Transferring the origin of co-ordinates to x_0, y_0 (Fig. 3(*a*)), and using polar co-ordinates, the stresses acting on a particle bounded by the arcs of two concentric circles and by two radial lines (Fig. 3(*b*)) are given by:

$$s_\rho = \frac{p}{6\,K^3}\Bigg[3\,\rho\,\sin\theta + 6\,K\,y_0\,\sin 2\,\theta + 3\,\rho\,\sin 3\,\theta + \left(\frac{3\,\rho K^3}{2} - \frac{3\,K\,\rho}{2}\right)\cos\theta + \left(6\,y_0 - 3\,K\,x_0 - 3\,K^3\,x_0\right)\cos 2\,\theta - \left(\frac{3\,\rho}{2}K^3 + \frac{9\,\rho}{2}K\right)\cos 3\,\theta + 3\,K^3\,x_0 + 6\,y_0 - 3\,K\,x_0\Bigg] \quad \text{....(10)}$$

$$s_s = \frac{p}{6\,K^3}\left[\left(\frac{3\,\rho}{2}\,K - \frac{3\,\rho}{2}\,K^3\right)\sin\theta + \left(6\,y_0 - 3\,K^3\,x_0 - 3\,K\,x_0\right)\sin 2\,\theta - \left(\frac{9\,K\,\rho}{2} + \frac{3\,K^3\,\rho}{2}\right)\sin 3\,\theta + 3\,\rho\cos\theta - 6\,K\,y_0\cos 2\,\theta - 3\,\rho\cos 3\,\theta\right] \quad (11)$$

and,

$$s_\theta = -\frac{p}{6\,K^3}\left[6\,K^3\left(\rho\cos^3\theta + x_0\cos^2\theta\right) + 12\left(\rho\sin^3\theta + y_0\sin^2\theta\right) - 6\,K\left(3\,\rho\sin^2\theta\cos\theta + 2\,y_0\sin\theta\cos\theta + x_0\sin^2\theta\right)\right] \quad (12)$$

in which ρ and θ are the polar co-ordinates.

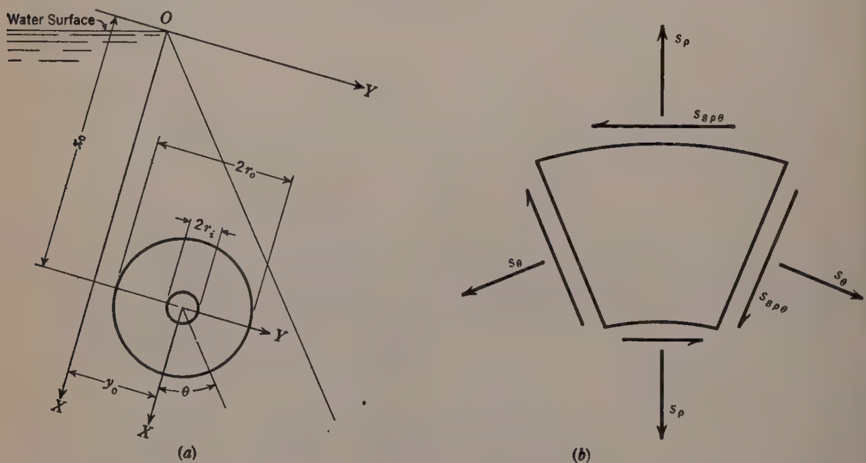

FIG. 3

If a solid disk of radius, $\rho = r_o$ (outer radius), is considered to be cut from the triangular section, the stresses acting on its boundary are obtained by inserting $\rho = r_o$ in Equations (10) and (11). If, now, $\rho = r_i$ (inner radius) is the radius of the circular opening, then in accordance with the assumption that the stresses on the boundary, $\rho = r_o$, are the same as if no hole existed, the problem reduces to the study of an annular closed ring subjected to the following stresses:

$$s_{\rho o} = A_0 + \sum_{n=1}^{n=3} A_n\cos n\,\theta + \sum_{n=1}^{n=3} B_n\sin n\,\theta \quad (13a)$$

$$s_{\rho i} = 0 \quad (13b)$$

$$s_{so} = C_0 + \sum_{n=1}^{n=3} C_n\cos n\,\theta + \sum_{n=1}^{n=3} D_n\sin n\,\theta \quad (13c)$$

and,

$$s_{si} = 0 \quad \text{.......(13}d\text{)}$$

in which $s_{\rho o}$ = the radial stress for a radius, r_o; $s_{\rho i}$ = the radial stress for a radius, r_i; s_{so} = the shear stress for a radius, r_o; s_{si} = the shear stress for a radius, r_i; and A_o, C_o, A_n, B_n, C_n, and D_n are defined as:

$$A_o = -\frac{p}{2\,K^3}\,(K^3\,x_o + 2\,y_o - K\,x_o) \quad \text{.......(14}a\text{)}$$

$$A_1 = -\frac{p\,r_o}{4\,K^3}\,(K^3 - K) = D_1 \quad \text{.......(14}b\text{)}$$

$$A_2 = -\frac{p}{2\,K^3}\,(2\,y_o - K\,x_o - K^3\,x_o) = -D_2 \quad \text{.......(14}c\text{)}$$

$$A_3 = \frac{p\,r_o}{4\,K^3}\,(K^3 + 3\,K) = -D_3 \quad \text{.......(14}d\text{)}$$

$$B_1 = -\frac{p\,r_o}{2\,K^3} = B_3 = C_3 = -C_1 \quad \text{.......(14}e\text{)}$$

$$B_2 = -\frac{p\,y_o}{K^2} = C_2 \quad \text{.......(14}f\text{)}$$

and,

$$C_o = 0 \quad \text{.......(14}g\text{)}$$

The analysis of this annular ring, given in Appendix II, results in the following formulas for the radial, circumferential, and shear stresses in the proximity of the opening:

$$\begin{aligned} s_\rho = {} & A_o\left(1 - \frac{r^2_h}{\rho^2}\right) + r_h\left(M_1 \cos\theta + M_2 \sin\theta\right)\left(\frac{\rho}{r_h} - \frac{r^3_h}{\rho^3}\right) \\ & - \left(A_2 \cos 2\,\theta + B_2 \sin 2\,\theta\right)\left(4\,\frac{r^2_h}{\rho^2} - 1 - 3\,\frac{r^4_h}{\rho^4}\right) \\ & - r_h\left(M_3 \cos 3\,\theta + M_2 \sin 3\,\theta\right)\left(-\frac{\rho}{r_h} + 5\,\frac{r^3_h}{\rho^3} - 4\,\frac{r^5_h}{\rho^5}\right) \end{aligned} \quad \text{.....(15)}$$

$$\begin{aligned} s_\theta = {} & A_o\left(1 + \frac{r^2_h}{\rho^2}\right) + r_h\left(M_1 \cos\theta + M_2 \sin\theta\right)\left(3\,\frac{\rho}{r_h} + \frac{r^3_h}{\rho^3}\right) \\ & - \left(A_2 \cos 2\,\theta + B_2 \sin 2\,\theta\right)\left(3\,\frac{r^4_h}{\rho^4} + 1\right) \\ & + r_h\left(M_3 \cos 3\,\theta + M_2 \sin 3\,\theta\right)\left(-\frac{\rho}{r_h} + \frac{r^3_h}{\rho^3} - 4\,\frac{r^5_h}{\rho^5}\right) \end{aligned} \quad \text{......(16)}$$

and,

$$s_s = r_h \left(\frac{\rho}{r_h} - \frac{r^3_h}{\rho^3}\right)\left(M_1 \sin\theta - M_2 \cos\theta\right) + \left(A_2 \sin 2\,\theta - B_2 \cos 2\,\theta\right)$$

$$\left(3\,\frac{r^4_h}{\rho^4} - 1 - 2\,\frac{r^2_h}{\rho^2}\right) + r_h\left(4\,\frac{r^5_h}{\rho^5} - \frac{\rho}{r_h} - 3\,\frac{r^3_h}{\rho^3}\right)\left(M_3 \sin 3\,\theta - M_2 \cos 3\,\theta\right) \text{.(17)}$$

in which $M_1 = -\frac{p}{4\,K^3}\left(K^3 - K\right)$; $M_2 = -\frac{p}{2\,K^3}$; and, $M_3 = \frac{p}{4\,K^3}\left(K^3 + 3\,K\right)$.

Equations (15), (16), and (17) suffice for the determination of the principal stresses near the opening. The hoop stresses may be determined from Equation (16) using $\rho = r_h$ and results in,

$$s_\theta = 2\,A_0 + 4\,r_h\,(M_1 \cos\theta + M_2 \sin\theta) - 4\,(A_2 \cos 2\,\theta + B_2 \sin 2\,\theta)$$

$$- 4\,r_h\,(M_3 \cos 3\,\theta + M_2 \sin 3\,\theta) \text{..............(18)}$$

Stresses Due to Mass

The equations of equilibrium of a particle (Fig. 3(*b*)) under the action of gravity forces are as follows:

$$\frac{\partial s_\rho}{\partial \rho} + \frac{\partial s_s}{\rho\,\partial\theta} + \frac{s_\rho - s_\theta}{\rho} + c\cos(\theta - \beta) = 0 \text{ ....(19a)}$$

and,

$$\frac{\partial s_\theta}{\rho\,\partial\theta} + \frac{\partial s_s}{\partial\rho} + \frac{2\,s_s}{\rho} - c\sin(\theta - \beta) = 0 \text{ ....(19b)}$$

in which c = the weight per unit volume; and β = the angle between the direction of gravity and the X-axis.

If the stresses are now defined by,

$$s_\rho = \frac{\partial^2 F}{\rho^2\,\partial\,\theta^2} + \frac{1}{\rho}\,\frac{\partial F}{\partial\rho} - c\,\rho\cos(\theta - \beta) \text{ ....(20a)}$$

$$s_\theta = \frac{\partial^2 F}{\partial\rho^2} - c\,\rho\cos(\theta - \beta) \text{ ....(20b)}$$

and,

$$s_s = -\frac{\partial}{\partial\rho}\left(\frac{\partial F}{\rho\,\partial\,\theta}\right) \text{ ....(20c)}$$

the equations of equlibrium and the compatibility resulting from Hooke's law will be satisfied if F is a solution of Equation (21):

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2 F}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial F}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2 F}{\partial\theta^2}\right) = 0 \text{ ...(21)}$$

It will be found that the Airy function, F, corresponding to mass and satisfying the boundary conditions for a triangular section is given by,

$$F_g = \frac{1}{6}\left[c_1 x^3 + (c_1 k_2 + c_2 - 2 c_2 k^2_2) y^3 + 3 c_2 k_2 x y^2\right] \ldots\ldots. (22)$$

in which, $c_1 = c \cos \beta$; $c_2 = c \sin \beta$; $k_2 = \cot \alpha$ (see Fig. 2). Transferring the origin to x_0, y_0, and using the definitions stated by Equations (20), the stresses are the following:

$$s_\rho = \frac{1}{6}\left[\left\{\frac{3 \rho c_1}{2} + \frac{3 \rho}{2} c_2 k_2 - 6 c_1 \rho\right\} \cos \theta \right.$$

$$+ \left\{3 c_2 k_2 x_0 + 3 y_0 (c_1 k_2 + c_2 - 2 c_2 k^2_2) - 3 c_1 x_0\right\} \cos 2\theta$$

$$+ \left\{\frac{9 \rho}{2} c_2 k_2 - \frac{3 \rho}{2} c_1\right\} \cos 3\theta + \left\{\frac{3 \rho}{2} (c_1 k_2 + c_2 - 2 c_2 k^2_2) - 6 c_2 \rho\right\} \sin \theta$$

$$- 6 c_2 k_2 y_0 \sin 2\theta + \frac{3 \rho}{2} (c_1 k_2 + c_2 - 2 c_2 k^2_2) \sin 3\theta$$

$$\left. + \left\{3 y_0 (c_1 k_2 + c_2 - 2 c_2 k^2_2) + 3 c_2 k_2 x_0 - 3 c_1 x_0 - 6 y_0 c_2\right\}\right] \ldots (23)$$

$$s_\theta = \frac{1}{6}\left[\left\{\frac{9 \rho}{2} c_1 + \frac{9 \rho}{2} c_2 k_2 - 6 c_1 \rho\right\} \cos \theta \right.$$

$$+ \left\{3 c_1 x_0 - 3 c_2 k_2 x_0 - 3 y_0 (c_1 k_2 + c_2 - 2 c_2 k^2_2)\right\} \cos 2\theta$$

$$+ \left\{\frac{3 \rho}{2} c_1 - \frac{9}{2} c_2 k_2 \rho\right\} \cos 3\theta + \left\{\frac{9 \rho}{2} (c_1 k_2 + c_2 - 2 c_2 k^2_2) - 6 c_2 \rho\right\} \sin \theta$$

$$+ 6 c_2 k_2 y_0 \sin 2\theta - \frac{3 \rho}{2} (c_1 k_2 + c_2 - 2 c_2 k^2_2) \sin 3\theta$$

$$\left. + \left\{3 y_0 (c_1 k_2 + c_2 - 2 c_2 k^2_2) + 3 c_2 k_2 x_0 - 3 c_1 x_0 - 6 y_0 c_2\right\}\right] \ldots (24)$$

and,

$$s_s = -\frac{1}{6}\left[\frac{3 \rho}{2}\left(c_1 k_2 + c_2 - 2 c_2 k^2_2\right) \cos \theta + 6 y_0 c_2 K_2 \cos 2\theta \right.$$

$$- \left\{\frac{3 \rho}{2}\left(c_1 k_2 + c_2 - 2 c_2 k^2_2\right)\right\} \cos 3\theta + \left\{\frac{9 \rho c_1}{2} - \frac{3}{2} k_2 c_2 \rho - 6 \rho c_1\right\} \sin \theta$$

$$+ \left\{3 c_2 k_2 x_0 + \left(c_1 k_2 + c_2 - 2 c_2 k^2_2\right) 3 y_0 - 3 x_0 c_1\right\} \sin 2\theta$$

$$\left. + \left\{\frac{9 \rho}{2} c_2 k_2 - \frac{3 \rho}{2} c_1\right\} \sin 3\theta\right] \ldots\ldots\ldots\ldots\ldots\ldots (25)$$

The stresses given in Equations (23), (24), and (25) may be broken into two separate stress systems each of which are compatible and satisfy Equation (21). Let $s_\rho = (s_\rho)_{\text{I}} + (s_\rho)_{\text{II}}$; $s_\theta = (s_\theta)_{\text{I}} + (s_\theta)_{\text{II}}$; and $s_S = (s_S)_{\text{I}} + (s_S)_{\text{II}}$. The stresses defined by the subscript, II, or the stresses in System II, are given by:

$$(s_\rho)_{\text{II}} = - c_1\,\rho \cos\theta - c_2\,\rho \sin\theta \ldots\ldots (26a)$$

$$(s_\theta)_{\text{II}} = - c_1\,\rho \cos\theta - c_2\,\rho \sin\theta \ldots\ldots (26b)$$

and,

$$(s_S)_{\text{II}} = 0 \ldots\ldots (26c)$$

The stresses of System I are then determined by: $(s_\rho)_{\text{I}} = s_\rho - (s_\rho)_{\text{II}}$; $(s_\theta)_{\text{I}} = s_\theta - (s_\theta)_{\text{II}}$; and, $(s_S)_{\text{I}} = s_S$.

Stresses Due to System I.—From this system,

$$A_0 = \frac{y_0}{2}\left(c_1 k_2 - 2\,c_2 k^2_2 - c_2\right) + \frac{x_0}{2}\left(c_2 k_2 - c_1\right) \ldots\ldots (27a)$$

$$A_1 = D_1 = \frac{r_o}{4}\left(c_1 + c_2 k_2\right) \ldots\ldots (27b)$$

$$A_2 = - D_2 = \frac{1}{2}\left\{x_o\left(c_2 k_2 - c_1\right) + y_0\left(c_1 k_2 + c_2 - 2\,c_2 k^2_2\right)\right\} \ldots (27c)$$

$$A_3 = - D_3 = \frac{r_o}{4}\left(3\,c_2 k_2 - c_1\right) \ldots\ldots (27d)$$

$$B_1 = - C_1 = \frac{r_o}{4}\left(c_1 k_2 + c_2 - 2\,c_2 k^2_2\right) \ldots\ldots (27e)$$

$$B_2 = C_2 = - c_2 k_2 y_0 \ldots\ldots (27f)$$

and,

$$C_0 = 0 \ldots\ldots (27g)$$

Using Equation (43), Appendix II, the state of stress around the opening due to System I is described by the following:

$$\begin{aligned} s_\rho = {} & A_0\left(1 - \frac{r^2_h}{\rho^2}\right) + \frac{r_h}{4}\left[\left(c_1 + c\ k_2\right)\cos\theta + \left(c_1 k_2 + c_2 - 2\,c_2 k^2_2\right)\sin\theta\right] \\ & \times\left(\frac{\rho}{r_h} - \frac{r^3_h}{\rho^3}\right) - \left(A_2 \cos 2\,\theta + B_2 \sin 2\,\theta\right)\left(4\,\frac{r^2_h}{\rho^2} - 1 - 3\,\frac{r^4_h}{\rho^4}\right) \\ & - \frac{r_h}{4}\left[\left(3\,c_2 k_2 - c_1\right)\cos 3\,\theta + \left(c_1 k_2 + c_2 - 2\,c_2 k^2_2\right)\sin 3\,\theta\right] \\ & \times\left[-\frac{\rho}{r_h} + 5\,\frac{r^3_h}{\rho^3} - 4\,\frac{r^5_h}{\rho^5}\right] \ldots\ldots (28) \end{aligned}$$

$$s_\theta = A_0\left(1+\frac{r^2_h}{\rho^2}\right)+\frac{r_h}{4}\left[\left(c_1+c_2 k_2\right)\cos\theta+\left(c_1 k_2+c_2-2 c_2 k^2_2\right)\sin\theta\right]$$
$$\times\left(3\frac{\rho}{r_h}+\frac{r^3_h}{\rho^3}\right)-\left(A_2\cos 2\theta+B_2\sin 2\theta\right)\left(3\frac{r^4_h}{\rho^4}+1\right)$$
$$+\frac{r_h}{4}\left[\left(3 c_2 k_2-c_1\right)\cos 3\theta+\left(c_1 k_2+c_2-2 c_2 k^2_2\right)\sin 3\theta\right]$$
$$\times\left(-\frac{\rho}{r_h}+\frac{r^3_h}{\rho^3}-4\frac{r^5_h}{\rho^5}\right)\dots\dots\dots(29)$$

and,

$$s_s = \frac{r_h}{4}\left[\left(c_1+c_2 k_2\right)\sin\theta+\left(c_1 k_2+c_2-2 c_2 k^2_2\right)\cos\theta\right]\left(\frac{\rho}{r_h}-\frac{r^3_h}{\rho^3}\right)$$
$$+\left(A_2\sin 2\theta-B_2\cos 2\theta\right)\left(3\frac{r^4_h}{\rho^4}-1-2\frac{r^2_h}{\rho^2}\right)$$
$$+\frac{r_h}{4}\left[\left(3 c_2 k_2-c_1\right)\sin 3\theta-\left(c_1 k_2+c_2-2 c_2 k^2_2\right)\cos 3\theta\right]$$
$$\times\left[4\frac{r^5_h}{\rho^5}-\frac{\rho}{r_h}-3\frac{r^3_h}{\rho^3}\right]\dots\dots\dots(30)$$

Substituting $\rho = r_h$ in Equation (29) the hoop stresses are:

$$s_\theta = 2A_0 + r_h\left[\left(c_1+c_2 k_2\right)\cos\theta+\left(c_1 k_2+c_2-2 c_2 k^2_2\right)\sin\theta\right]$$
$$-4\left(A_2\cos 2\theta+B_2\sin 2\theta\right)$$
$$-r_h\left[\left(3 c_2 k_2-c_1\right)\cos 3\theta+\left(c_1 k_2+c_2-2 c_2 k^2_2\right)\sin 3\theta\right]\dots(31)$$

Hoop Stresses Due to System II.—An investigation of System II shows that these stresses hold the weight of a solid disk of radius r_o, in equilibrium, whereas the stresses of System I give a zero resultant on the circumference, $\rho = r_o$. Consider an annular disk of the outer radius, r_o, and the inner radius, r_i, subjected to gravity forces. If all the stresses on the radius, r_i are zero, the resultant of all the forces on the radius, r_o, must be equal to the weight of the disk. If System II is to be applied to an annular disk, c must be made to equal to:

$$c = -c\left(\frac{r^2_i}{r^2_o}-1\right)\dots\dots\dots(32)$$

resulting in a modified System II given by $(s_r)_{II} = c\left(\frac{r^2_i}{r^2_o}-1\right)\rho\cos\theta$; $(s_\theta)_{II} = c\left(\frac{r^2_i}{r^2_o}-1\right)\rho\cos\theta$; and $(s_s)_{II} = 0$.

An Airy function, F, may be written for this system:

$$F = \tfrac{1}{2}\, c\, r_i^2\, \rho\, \theta \sin \theta \text{.......................(33)}$$

From Equation (20) it follows that,

$$s_\rho = c \left(\frac{r_i^2}{\rho} - \rho \right) \cos \theta \text{..................(34a)}$$

$$s_\theta = -\, c\, \rho \cos \theta \text{.........................(34b)}$$

and,

$$s_s = 0 \text{..............................(34c)}$$

For $\rho = r_i$, $s_\rho = s_s = 0$; and, for $\rho = r_o$, $s_\rho = c \left(\frac{r_i^2}{r_o} - r_o \right) \cos \theta$; and, $s_s = 0$.

The hoop stresses due to System II are $s_\theta = c\, r_h \cos \theta$. These stresses must be added to those of System I to describe the complete state of stress due to mass.

Application.—The foregoing equations will be applied to the determination of the stresses around a circular hole in a buttress of a hollow type dam with the following design characteristics:

Clear spacing between buttresses, in feet................	55
Average thickness of buttress, in feet..................	5
Up-stream slope of buttress...........................	0.9
Down-stream slope of buttress.........................	0.36
Height of dam, h, in feet.............................	200
Radius of circular opening, 0.02 h, in feet.............	4
Weight of water, in pounds per cubic foot..............	62.5
Weight of concrete, c, in pounds per cubic foot..........	150

The location of the center of the opening is given by $x_o = 0.5h$, and $y_o = 0.2h$. From the spacing, thickness, and up-stream slope of the buttress, $p = \frac{(62.5)\,(55 + 5)\,(\sin 48^\circ)}{5} = 556$ lb per sq. ft.; and $c = p \times \frac{150}{556} = 0.27\, p$. It follows that: $K = \tan \alpha = 1.864$; $K_2 = \cot \alpha = 0.5365$; and, $\beta = 42$ degrees.

Stresses Due to Water Load.—Using Equations (14):

$A_0 = -\,0.2089\, p\, h$	$M_1 = -\,0.1780\, p$
$A_2 = +\,0.29\, p\, h$	$M_2 = -\,0.07721\, p$
$B_2 = -\,05757\, p\, h$	$M_3 = +\,0.4657\, p$

Substituting in Equation (18) the "hoop stresses" due to water load are:

$$s_\theta = p\, h\, [-\,0.4178 - 0.01424 \cos \theta - 1.16 \cos 2\theta - 0.03726 \cos 3\theta$$
$$- 0.006177 \sin \theta + 0.2303 \sin 2\theta + 0.006177 \sin 3\theta] \text{.......(35)}$$

Stresses Due to Mass.—Let $c_1 = c \cos \beta = 0.7431\ c$; and, $c_2 = c \sin \beta = 0.6691\ c$. From Equations (27): $A_0 = -0.1616\ c\ h$; $A_2 = -0.02780\ c\ h$; and, $B_2 = -0.07180\ c\ h$.

The "hoop stresses" due to mass are combined with System II:

$$s_\theta = c\,h \Big[-0.3232 + 0.00204 \cos \theta + 0.1112 \cos 2\theta - 0.006678 \cos 3\theta + 0.01365 \sin \theta + 0.2872 \sin 2\theta - 0.01365 \sin 3\theta \Big] \ldots\ldots\ldots(36)$$

Combining Equations (35) and (36) the "hoop stresses" due to water and mass are, when $c = 0.27\ p$, as follows:

$$s_\theta = p\,h \Big[-0.5051 - 0.00829 \cos \theta\ 1.13 \cos 2\theta - 0.03906 \cos 3\theta - 0.00249 \sin \theta + 0.3078 \sin 2\theta + 0.00249 \sin 3\theta \Big] \ldots\ldots\ldots(37)$$

TABLE 1.—HOOP STRESSES DUE TO COMBINED MASS AND WATER LOADS ($p\,h = 771$ lb per in.^{2})

− = COMPRESSION			+ = TENSION		
θ	s_θ	Unit stress, in pounds per square inch	θ	s_θ	Unit stress, in pounds per square inch
0	—1.6825 ph	—1300	180	1.5877 ph	—1221
30	—0.8094 ph	—623	210	—0.7976 ph	—613
60	+0.3583 ph	+276	240	+0.2937 ph	+226
90	+0.6200 ph	+478	270	+0.6300 ph	+485
120	—0.2438 ph	—188	300	—0.1696 ph	—131
150	—1.3282 ph	—1022	330	—1.3451 ph	—1035

These stresses are plotted in Fig. 4 and the unit stresses are given in Table 1. Maximum tensile stresses exist in the neighborhood of $\theta = 90°$ and $\theta = 270$ degrees. For $\theta = 90°$, the circumferential stress, s_θ, as a function of $\left(\frac{\rho}{r_h}\right)$ is given by;

$$s_\theta = p\,h \Big[-0.2525 \left(1 + \frac{r^2_h}{\rho^2}\right) - 0.000622 \left(3\,\frac{\rho}{h} + \frac{r^3_h}{\rho^3}\right) + 0.2825 \left(3\frac{r^4_h}{\rho^4} + 1\right) \times 0.000622 \left(-\frac{\rho}{r_h} + \frac{r^3_h}{\rho^3} - 4\,\frac{r^5_h}{\rho^5}\right) \Big] \ldots\ldots\ldots\ldots\ldots\ldots(38)$$

These stresses are plotted in Fig. 5.

To determine the total tension which must be taken by the reinforcing steel the area under the curve of Fig. 5 may be determined, or the following integral evaluated:

$$\text{Total tension} = \int_{r_h}^{2.5 r_h} s_\theta\, d\rho \ldots\ldots\ldots\ldots\ldots\ldots(39)$$

in which s_θ as given by Equation (38) is to be inserted in Equation (39). The upper limit is determined by an examination of Fig. 5. Performing the integration: Total tension $= ph\,r_h\,[-0.2525\ (2.1) - 0.000622\ (8.295) + 0.2825 \times 2.436 + 0.000622\ (-\ 3.1794)] = 68\,988$ lb. Using an average stress of

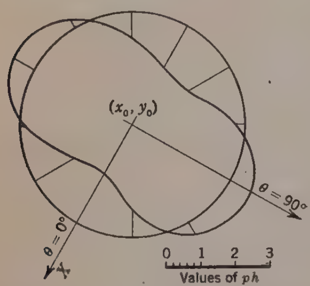

FIG. 4.—s_θ STRESSES DUE TO MASS AND WATER

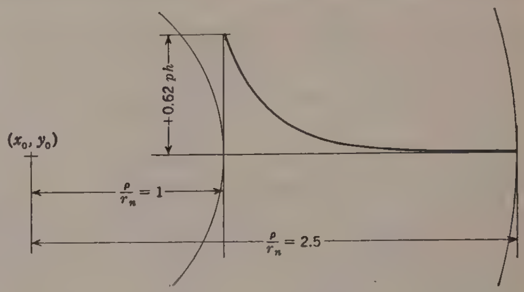

FIG. 5.—VARIATION OF s_θ STRESS ALONG SECTION $\theta = 90$ DEGREES

18 000 lb per sq in., the area of steel required is $\dfrac{68\,988}{18\,000} = 3.83$ in.² The spacing of this steel is easily determined from a consideration of Fig. 5. The percentage of steel is $\dfrac{3.83}{(1.5\ r_h)\ (144)} = 0.444$ per cent. These values are for a unit thickness of buttress.

CONCLUSIONS

The formulas derived in this paper enable the designer to determine the regions around openings in dams and buttresses where tensile stresses may exist. Stress trajectories may also be determined by use of these equations. Furthermore, an estimate may be obtained of the reinforcement needed to take care of such tensile stress. In the case of gravity dams these equations are limited to longitudinal galleries.

APPENDIX I

NOTATION

In the following notation, presented for convenience of reference, an effort has been made to conform as nearly as practicable with "Symbols for Mechanics, Structural Engineering, and "Testing Materials", advanced by the American Standards Association.[2] Symbols used simply as general mathematical coefficients are not included in the list, their definitions being given clearly in the text. Where such coefficients conflict with symbols also used in the paper (such as c = mass), the distinction is explained at each point.

C = a constant;
c = weight per unit volume;
F = the Airy stress function;

f = a substitution factor defined by Equations (47*d*) and (47*h*), Appendix II;

G = a substitution factor defined by Equation (46*b*), Appendix II;

g = the gravity constant; also, where so indicated, g_n = a substitution factor defined by Equations (47)*e*) and (47*i*), Appendix II;

h = height; also, where so indicated, h_n = a substitution factor defined by Equations (47*f*) and (47*j*), Appendix II;

J = a substitution factor defined by Equation (47*c*), Appendix II;

K = $\tan \alpha$ = a substitution factor; $K_2 = \cot \alpha$;

k_n = a substitution factor defined by Equations (47*g*) and (47*k*), Appendix II;

$\left.\begin{matrix} M_1 \\ M_2 \\ M_3 \end{matrix}\right\}$ = (see Equations (48), (49), and (50), Appendix II);

n = an abstract number;

p = unit, hydrostatic water pressure;

q = the ratio, $\dfrac{r_i}{r_o}$;

r = radius of a circle, as distinct from the polar co-ordinate distance, ρ; r_h = radius of a circular hole; r_i = the inner radius of an annular ring; r_o = outer radius of an annular ring;

s = unit stress in general; in particular, s = total axial load divided by gross area; s_n = normal stress; s_s = shear stress; s_x = stress in an X-direction; s_y = stress in a Y-direction; s_ρ = stress in a radial plane; s_θ = stress perpendicular to a radial plane at any angle, θ; $s_{\rm I}$ = stresses defined as belonging to System I; $s_{\rm II}$ = stresses defined as belonging to System II;

x = distance measured in an X-direction; x_0 = the x-co-ordinate of center of hole;

y = distance measured in a Y-direction; y_0 = the y-co-ordinate of center of hole;

α = angle between the battered faces of a dam.

θ = angular distance; a polar co-ordinate;

ρ = a radial distance; polar co-ordinate;

APPENDIX II

Analysis of Annular Ring

An annular ring of outer radius, r_o, inner radius, r_i subjected to the boundary stresses as given by Equations (13) can best be studied by use of Airy's function in polar co-ordinates. Corresponding to Equation (7) in cartesian co-ordinates, there is a similar equation in polar co-ordinates:

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2\phi}{\partial\rho^2} + \frac{1}{\rho}\,\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\,\frac{\partial^2\phi}{\partial\theta^2}\right) = 0 \quad \ldots (40)$$

with stresses defined as follows:

$$\left.\begin{aligned} s_\rho &= \frac{1}{\rho}\frac{\partial\phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\phi}{\partial\rho^2} \\ s_\theta &= \frac{\partial^2\phi}{\partial\theta^2} \\ \text{and,} \quad s_s &= -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\phi}{\partial\theta}\right) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(41)$$

The general solution[5] of Equation (40) is:

$$\begin{aligned} \phi &= a_0 \log\rho + b_0\rho^2 + c_0\rho^2\log\rho + d_0\rho^2\theta + a'_0\theta \\ &+ \frac{a_1}{2}\rho\,\theta\sin\theta + \left(b_1\rho^3 + a'_1\rho^{-1} + b'_1\rho\log\rho\right)\cos\theta \\ &- \frac{c_1}{2}\rho\,\theta\cos\theta + \left(d_1\rho^3 + c'_1\rho^{-1} + d'_1\rho\log\rho\right)\sin\theta \\ &+ \sum_{n=2}^{n=\infty}\left(a_n\rho^n + b_n\rho^{n+2} + a'_n\rho^{-n} + b'_n\rho^{-n+2}\right)\cos n\theta \\ &+ \sum_{n=2}^{n=\infty}\left(c_n\rho^n + d_n\rho^{n+2} + c'_n\rho^{-n} + d'\rho^{-n+2}\right)\sin n\theta \ldots\ldots(42) \end{aligned}$$

in which a_n, b_n, c_n, and d_n are mathematical constants. In order that the stress distribution and displacements may be single valued, $c_0 = d_0 = 0$; and in order to satisfy the boundary conditions imposed by Equations (13):

$$a_0 = -\frac{r_o^2\,q^2}{1-q^2}A_0 \ldots\ldots\ldots\ldots(43a)$$

and,

$$b_0 = \frac{1}{2}\frac{A_0}{1-q^2} \ldots\ldots\ldots\ldots(43b)$$

From the fact that $A_1 = D_1$ and $B_1 = -C_1$, it follows that $d'_1 = b'_1 = 0$. In their complete study of the circular ring, Messrs. Coker and Filon[6], have shown that the remaining constants are given by,

$$b_1\ r_o = \frac{A_1}{2(1-q^4)} \ldots\ldots\ldots\ldots(44a)$$

$$a'_1 = \frac{A_1 r_o^3 q^4}{2(1-q^4)} \ldots\ldots\ldots\ldots(44b)$$

$$d_1\ r_o = \frac{B_1}{2(1-q^4)} \ldots\ldots\ldots\ldots(44c)$$

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[5] "Theory of Elasticity", S. Timoshenko, p. 114.

[6] "Photo-Elasticity", by E. G. Coker and L. N. G. Filon, 1931, pp. 374 *et seq.*

and,

$$c'_1 = \frac{B_1 r^3_o q^4}{2 (1 - q^4)} \quad \text{(44}d\text{)}$$

When n is equal to, or greater than, 2:

$$b_n \; r_o^n = Q_n f_n \quad G_n g_n \quad \text{(45}a\text{)}$$

$$a_n \; r_o^{n-2} = Q_n g_n - J_n f_n \quad \text{(45}b\text{)}$$

$$a'_n \; r_o^{-n} \; b^{-2} = Q_n h_n - G_n k_n \quad \text{(45}c\text{)}$$

$$b'_n \; r_o^{-n} = Q_n k_n - J_n h_n \quad \text{(45}d\text{)}$$

$$\mathrm{d}_n \; \mathrm{r}_o^n = Q_n f'_n - G_n g'_n \quad \text{(45}e\text{)}$$

$$c_n \; r_o^{n-2} = Q_n g'_n - J_n f'_n \quad \text{(45}f\text{)}$$

$$c'_n \; r_o^{-n} \; b^{-2} = Q_n h'_n - G_n k'_n \quad \text{(45}g\text{)}$$

and,

$$d'_n \; r_o^{-n} = Q_n k'_n - J_n h'_n \quad \text{(45}h\text{)}$$

in which,

$$Q_n = \frac{1 - q^{2n}}{2\,n\,[(1 - q^{2n})^2 - n^2 q^{2n-2} (1 - q^2)^2]} \quad \text{(46}a\text{)}$$

$$G_n = \frac{1 - q^{2n-2}}{2\,(n + 1)\,[(1 - q^{2n})^2 - n^2 q^{2n-2} (1 - q^2)^2]} \quad \text{(46}b\text{)}$$

$$J_n = \frac{1 - q^{2n+2}}{2\,(n - 1)\,[(1 - q^{2n})^2 - n^2 q^{2n-2} (1 - q^2)^2]} \quad \text{(46}c\text{)}$$

$$f_n = n\,(A_n - D_n) \quad \text{(46}d\text{)}$$

$$g_n = n\,A_n - (n + 2)\,D_n \quad \text{(46}e\text{)}$$

$$h_n = -\,q^{2n-2}\,(n\,A_n + (n - 2)\,D_n) \quad \text{(46}f\text{)}$$

$$k_n = -\,n\,q^{2n}\,(A_n + D_n) \quad \text{(46}g\text{)}$$

$$f'_n = n\,(B_n + C_n) \quad \text{(46}h\text{)}$$

$$g'_n = n\,B_n + (n + 2)\,C_n \quad \text{(46}i\text{)}$$

$$h'_n = -\,q^{2n-2} \left\{ n\,B_n - (n - 2)\,C_n \right\} \quad \text{(46}j\text{)}$$

and,

$$k'_n = -\,n\,q^{2n} \left\{ B_n - C_n \right\} \quad \text{(46}k\text{)}$$

Substituting values of A_n in Equations (14) into Equations (46), the factors are determined to be as listed in Table 2. These values can be used

TABLE 2.—LIST OF FACTORS IN EQUATIONS (46), IN TERMS OF A_n AND B_n

Values of n	f	f'	g	g'	h	h'	k	k'
2	$4A_2$	$4B_2$	$6A_2$	$6B_2$	$-2q^2A_2$	$-2q^2B_2$	0	0
3	$6A_3$	$6B_3$	$8A_3$	$8B_3$	$-2q^4A_3$	$-2q^4B_3$	0	0

for the solution of Equations (45). Assuming that a plate is infinite so that $q = \frac{r_i}{r_o} = 0$, the stresses, s_ρ, s_θ, and s_s, can be obtained by the use of Equations (41) to (45), inclusive, the result being as follows:

$$s_\rho = A_0\left(1 - \frac{r^2_h}{\rho^2}\right) + r_h\left(M_1 \cos\theta + M_2 \sin\theta\right)\left(\frac{\rho}{r_h} - \frac{r^3_h}{\rho^3}\right)$$
$$- \left(A_2 \cos 2\theta + B_2 \sin 2\theta\right)\left(4\frac{r^2_h}{\rho^2} - 1 - 3\frac{r^4_h}{\rho^4}\right)$$
$$- r_h\left(M_3 \cos 3\theta + M_2 \sin 3\theta\right)\left(-\frac{\rho}{r_h} + 5\frac{r^3_h}{\rho^3} - 4\frac{r^3_h}{\rho^3}\right) \ldots\ldots(47)$$

$$s_\theta = A_0\left(1 + \frac{r^2_h}{\rho^2}\right) + r_h\left(M_1 \cos\theta + M_2 \sin\theta\right)\left(3\frac{\rho}{r_h} + \frac{r^5_h}{\rho^5}\right)$$
$$- \left(A_2 \cos 2\theta + B_2 \sin 2\theta\right)\left(3\frac{r^4_h}{\rho^4} + 1\right)$$
$$+ r_h\left(M_3 \cos 3\theta + M_2 \sin 3\theta\right)\left(-\frac{\rho}{r_h} + \frac{r^3_h}{\rho^3} - 4\frac{r^5_h}{\rho^5}\right) \ldots\ldots\ldots(48)$$

and,

$$s_s = r_h\left(\frac{\rho}{r_h} - \frac{r^3_h}{\rho^3}\right)\left(M_1 \sin\theta - M_2 \cos\theta\right)$$
$$+ \left(3\frac{r^4_h}{\rho^4} - 1 - 2\frac{r^2_h}{\rho^2}\right)\left(A_2 \sin 2\theta - B_2 \cos 2\theta\right)$$
$$+ r_h\left(4\frac{r^5_h}{\rho^5} - \frac{\rho}{r_h} - 3\frac{r^3_h}{\rho^3}\right)\left(M_3 \sin 3\theta - M_2 \cos 3\theta\right) \ldots\ldots\ldots(49)$$

in which $M_1 = -\frac{p}{4K^3}\left(K^3 - K\right)$; $M_2 = -\frac{p}{2K^3}$; and $M_3 = \frac{p}{4K^3}\left(K^3 + 3K\right)$.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

PAPERS

RECLAMATION AS AN AID TO INDUSTRIAL AND AGRICULTURAL BALANCE

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Synopsis

The economic pattern that has evolved from the troublesome years of 1929 to 1936 indicates that the future industrial structure will be related intimately to agriculture. Reclamation projects scattered over the Western States can have a profound influence in strengthening this link between the factory and the soil because many of these districts are ideally suited for development as complete economic units rather than as agricultural projects alone. A review of the benefits that may be obtained by co-ordinating industrial and agricultural activities on reclamation projects is presented herein.

The writers have studied this problem from three angles: First, the decentralization of industry; second, the diversification of labor; and third, the co-ordination of industry and agriculture. Part I is devoted to a review of the basic principles of each of these factors. In Part II these principles are applied broadly to an actual project to illustrate the industrial and social growth that could follow the construction of hydraulic works in a given area. These potential benefits are illustrated further in Part III by a more detailed analysis which demonstrates the possibilities of co-ordinating industry and agriculture within one of the irrigation districts that will benefit from the projects. Although the conclusions are based on a specific investigation, they would also undoubtedly apply to many of the new irrigated developments that have been made available by Boulder Dam, and other great water conservation projects.

Note.— Discussion on this paper will be closed in the **March, 1937,** *Proceedings.*

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PART I.—BASIC PRINCIPLES

The Decentralization of Industry

Historical.—Industrial congestion in thickly populated cities began with the domestic system long before the industrial revolution. In the early days employers furnished tools, equipment, and housing to workers, who conducted most of the manufacturing operations within their own homes.

The domestic system led to the establishment of the manufacturing towns and cities and these, later, furnished the labor supply for the factories which sprang up during the industrial revolution. The first commercial steam engine manufactured by James Watt in 1776—the engine that was really to begin the industrial revolution—started factories on the road to overgrowth. Steam power had to be used at its source; therefore, great size and economy of operation went hand in hand.

A continued increase in the population densities of industrial cities accompanied the overgrown factories that followed the industrial revolution. Many of the wretched social conditions, that the nation is now striving so desperately to correct, have been inherited from this particular stage of economic evolution. The industrial worker of the city, with his abilities narrowed by mass-production methods, has no alternate means of livelihood during "shut-downs."

The mobility of electric power stands in sharp contrast to the fixed nature of steam power. Electric power, therefore, will be an important factor in the breaking down of great industrial units into smaller and more efficient production plants. Long-distance electrical transmission can make it possible to distribute these smaller plants over rural areas in locations where co-ordination with agriculture may be effected.

Principles and Trends.—Decentralization means the transfer of industry from the great factories of the industrial city to smaller plants in suburban or rural areas. In some cases both workers and plants are being relocated; in others, branch plants are being established.

The purposes of decentralization are: (1) To build partly self-sustaining areas; (2) through the establishment of controls, to balance production and consumption within these areas; (3) to lower greatly the cost of distribution; (4) to increase living standards by decreasing costs and by increasing both production and consumption; (5) to eliminate the evil effects of mass-production methods on industrial workers by the diversification of labor; and (6) to create better social conditions by moving workers and their families away from congested industrial districts and giving them the many advantages incidental to suburban or country life.

Industry is moving away from the congested industrial centers of the East. Evidence of this fact may be found in the "Industrial Census" which shows that the center of gravity of manufactures moved 329 miles westward between 1849 and 1919. During the thirty years between 1899 and 1929, in the areas of "primary concentration" (the big industrial cities), the number of wage jobs per 1 000 population fell from 124 to 106. In the areas of "secondary concentration" (the outlying fringes of the great manufacturing

centers and the smaller industrial towns), the corresponding decrease was less—from 106 to 105. Throughout the remainder of the United States, or the rural areas, however, the number actually increased from 34 to 45.

Other evidence of the trend of industry to locate in suburban or rural areas may be found in the statistics of plant movements between 1927 and 1929. During that period, 287 industrial plants, affording 18 599 wage jobs, were moved. Of this number 5 654 employees were moved away from the big cities, 1 933 going to the smaller cities and 3 721 going into the country or to small rural towns.

Future rural relocations will generally be accompanied by decreases in the sizes of production units. Recent studies of the economic sizes of manufacturing plants have shown conclusively that relatively small production units give the most satisfactory results. L. W. W. Morrow, Editor of the *Electrical World*, presented the following significant facts[3] relating to this subject.

An investigation of the machine tool industry made in 1932 shows that the economic size of a plant is one that uses 80 000 to 100 000 man-hr per yr; yet $96\frac{1}{2}\%$ of the plants in this industry have a greater productive capacity. Studies of three other industries show the same facts—that is, a relatively small production unit is most economical; but at present the greatest productive capacity is found in the larger plants.

In 1933 a certain manufacturer supplied the national market from three plants and manufactured eight products. To-day, this manufacturer has forty plants, one in each market area, and sells sixty-five products through one sales organization. Present earnings are highly satisfactory.

As a result of his studies, Mr. Morrow concluded that:

"An analysis of the growth of manufacturers shows that small plants or branch plants are being built in each market area and, on the whole, these are more prosperous than the large plants supplying the national market because they restrict their activities to sales in these local areas. The manufacturers who make one product for the national market with a single production plant in one location are suffering the greatest losses."

Mr. Henry Ford, a pioneer in the decentralization movement in the United States, has long advocated both the diversification of labor and a return to the small factory. Extensive experiments in industrial decentralization have been conducted in the vicinity of Dearborn, Mich., with the results summarized in the following statement by Mr. Ford[4]:

"Ten years ago we started seven village industries on small water power sites, all within twenty miles of Dearborn, our purpose being to combine the advantages of city wages with country living. The experiment has been a continuous success. Overhead cost has been less than in the big factory, and the workers would not hear of going back to the city shops."

Mr. Ford began to acquire land for this experiment several years ago. To-day (1936), his holdings for this purpose embrace 40 000 acres. Farms grouped in large tracts—the largest is 10 000 acres—are tilled on the co-opera-

[3] "Balancing Technocracy" *Electrical World*, January 14, 1933.

[4] "Farm and Factory," *New York Times*, June 3, 1932.

tive plan. Many Ford parts, including starters, lamps, gages, and drills are produced in fourteen small plants located in rural areas within fifty miles of Detroit, Mich. In 1934, these plants produced $8 000 000 worth of parts and tools, employed 2 500 workers, and paid $1 500 000 in wages to these small town workers.[5]

From the foregoing it is evident that decentralization is no longer an academic question for industry; experience has proved this to be a beneficial and constructive measure. No clearer picture of the possibilities of decentralization could be obtained than that given by the Secretary of Agriculture[6]:

"The ten million unemployed plus the five million living on land which can never be farmed are a continuing menace to the established industry and agriculture of the United States. To solve it means decentralized industrial planning relative to land. If the heads of our two hundred leading corporations were to take into account the full significance of paved roads, autos, trucks, high line electricity, and the increased happiness of human beings close to the land, might they not enthusiastically start a decentralized, industrial, self subsistence homestead program on a scale which would jerk us out of the depression for years to come"?

THE DIVERSIFICATION OF LABOR

Need of Diversification.—The rate of increase of unemployment has kept pace with technological progress. Each year occupational obsolescence has been sending increasing numbers of men to the "scrap heap." Consider, for example, the following unemployment statistics for the relatively prosperous years of 1923 to 1929, inclusive:

Year:	Unemployed:
1923	1 500 000
1925	1 775 000
1927	2 000 000
1929	3 000 000 to 5 000 000

The reasons for this steady increase in unemployment are suggested by the following tabulation[7], which shows, for certain commodities, the increases in production and the decreases in labor force for the years 1923 to 1927, inclusive:

Industry	Increase in production	Decrease in labor force
Oil refineries	84%	5%
Tobacco	53%	13%
Meat	20%	19%
Railroads	30%	1%
Building (Ohio only)	11%	15%
Coal	4%	15%
Steel	8%	9%
Cotton	3%	13%
Lumber	6%	21%

[5] *New York Times*, September 8, 1935.

[6] "New Frontiers," by the Hon. Henry A. Wallace.

[7] From "An Introduction to Problems of American Culture," by Professor Rugg, pp. 182 and 194.

Instances of unemployment due to technological improvements are legion and space permits only typical examples to be included herein. Improved mining methods have greatly increased the output per worker and, consequently, decreased the number of workers. During the past thirty years the output per worker in soft coal mining increased 60%, in copper mining, 100%, and in the mining of ore, 200 per cent. These increases in efficiency resulted in the permanent unemployment of 64 000 men, engaged in the extractive mineral industries, in the boom year of 1929[8].

The foregoing data present a better picture of occupational obsolescence than could be obtained from similar statistics for the depression years of 1930 to 1935. The abnormal situation in these years was influenced by so many factors that it would be impossible to segregate the effect of any single cause of unemployment.

Probably the most startling development in occupational obsolescence since 1929 was revealed in 1933 by the report of the Henderson Committee[9] in the automobile industry. It was stated therein that keen competition has speeded this industry to a pace considered too fast for men of forty and that, in some instances, 19 men do the work of 250 by contrast with 1929.

Other findings of the Henderson report were of an equally surprising nature; for example, in one plant a door is now made in two stamping operations. In 1929, a door manufactured by this same plant had twenty-six different parts and required several times as many men to make it. Body framing which cost $3.00 in 1929, now (1936) costs $0.30. It cost $0.60 to hang one door in 1929; it cost $0.9 to hang four doors in 1935. Body trimming cost $12.00 in 1929 it costs $4.00 to-day.

The Henderson Committee report stated:

"Less than five years ago a well known auto manufacturer finished 100 eight cylinder motor blocks on a given line-up with 250 men. To-day the same line-up finishes 250 motor blocks, with 20% more operations with only 19 men. Men were paid $13.20 per hundred blocks five years ago and through the use of tungsten carbide tool tips men now do the same work for $5.20."

It was shown that single machines introduced in factories are automatically removing from 100 to 250 employees.

Many other instances could be cited; the foregoing, however, will serve the purpose of this paper. Without turning to other illustrations of the present trend, it is obvious that two conclusions may be drawn regarding the effects of mass-production methods on a worker: First, his calling may be swept away almost without notice by technological changes, leaving him without alternate means of support; and second, mass-production methods are narrowing to such a degree that little or no mental labor is required. New industries, to absorb the workers thus eliminated from the progressive industries, will doubtless eventualy be evolved, but during their development the men and their families suffer greatly.

8 "America's Capacity to Produce," p. 155.

9 *New York Herald-Tribune*, February 7, 1935

The best quick remedy for this situation lies in the diversification of labor. Productive employment in work supplementary to that of the factory must be planned to fill the extended periods of idleness that have accompanied each depression. Although this need has been recognized for many years, it has been only recently that a few industrialists have taken steps to provide some alternate employment during the idle time.

Experiments in the Diversification of Labor.—Obviously, many kinds of work could be planned on a part-time basis. A return to the soil, however, offers the easiest and the most direct method of capitalizing idle hours. This fact was recognized by a group of progressive manufacturers who were trying every available means of coping with the unemployment problem during the depression. With a view, at least, toward keeping workers supplied with food, suitable tracts of land adjacent to the plants were purchased or leased and prepared for planting. Part-time and idle employees then cultivated these tracts and received in return for their labor either the produce of their individual plots or a proportionate share of the yield if the tracts were operated on the co-operative plan.

The experiences gained in administering these agricultural projects as supplements to industry bear a direct relation to this investigation. These experiences enabled the writers to base their analysis on actual operating data rather than on assumptions.

In giving consideration to agriculture as an alternate occupation for industrial employees, many questions arise at once. For example, what classes of labor (skilled or common) adapt themselves most readily to this plan? Would skilled workers care to divide their time between the farm and the factory? Assuming that these workers are inclined to try the experiment, would they be qualified by temperament to grasp an entirely new occupation? What is the cash outlay required of an industry to put such a plan in operation, and what are the returns? Is the worker compensated adequately for his hours of agricultural work? What time must be devoted to such work and how much land is required per dependent? What is the best plan of management? What educational measures are necessary?

In order to obtain answers to these questions the writers studied in detail the results obtained in eight experiments in industrial co-operative gardening. Space limitations permit only two of these to be described herein.

Example 1

General Plan.—During 1931 and 1932 more than 500 former workers of a certain company were unemployed and many others were working only part-time due to an unavoidable curtailment of production schedules. In order to minimize the monetary loss and the physical suffering caused by extensive lay-offs or part-time employment, the Company aligned itself with the municipal government and the civic relief organization. A co-operative farm plan was selected as the most practical solution of the problem.

Consideration was given to both the individual-plot and the mass-production methods of operation. Although no precedent was available for the

latter, the management decided that mass-production methods were equally applicable to gardening and manufacturing. The satisfactory results that followed the adoption of this policy are particularly significant.

Early in May, 1932, leases were secured on three tracts aggregating 275 acres in a river valley five miles from the factories. Of this land, 200 acres were tillable. The Employment Manager, was made Director of the project and an engineer with extensive agricultural and industrial experience, was engaged to prepare the detailed plans and to supervise the actual field operations. Plowing was begun as rapidly as the land could be cleared of brush and débris and made ready for cultivation. Drainage ditches, culverts, a small headquarters building, and sanitary facilities were then provided. By May 12, 2 000 early cabbage plants, 2 000 tomato plants, and several bushels of onions had been set out and these were followed immediately by the planting of other vegetables.

The labor-rotation plan, based on the individual working one 8-hr day per week, was followed. Schedules were based on an anticipated total of 150 000 man-hr, embracing 750 men over the 25-week period. Actually, a total of 936 men took an active part in the gardening. When all planting was completed the gardens had the acreage in crops shown by Table 1.

TABLE 1.—ACREAGE OF INDUSTRIAL CO-OPERATIVE GARDEN PROJECT, EXAMPLE 1

Crop (1)	Area, in acres (2)	Crop (1)	Area, in acres (2)	Crop (1)	Area, in acres (2)	Crop (1)	Area, in acres (2)
Potatoes	63.0	Bush and pole beans	8.1	Peppers	2.0	Onions	0.6
Navy beans	40.0	Turnips	7.6	Carrots	1.2	Swiss chard	0.5
Field corn	25.5	Beets	3.8	Kale	1.1	Radishes	0.4
Sweet corn	11.0	Pumpkins	3.0	Summer squash	1.0	Mustard	0.3
Tomatoes	9.5	Peas	3.0	Spinach	1.0	Lettuce	0.2
Cabbage	8.5						

Before planting the potatoes, consultations were held with the staff of the Experiment Station at the State University and with the champion potato grower of the State, a down-State farmer, and 1 300 bushels of certified seed potatoes were obtained from a reliable source.

Less than 25% of the workers had a true farm background. Although this general lack of gardening experience caused the work to progress rather slowly at first, the workers familiarized themselves with their new occupation rapidly. The instructions of the supervisor were grasped quickly and in a few weeks a fair degree of skill in farm work had been developed.

No records were kept of the percentages of the classes of labor (that is, skilled or unskilled), or of the relative performance of each class. It was found, however, that both classes readily adapted themselves to the program. The results of this experiment show conclusively that industrial workers can readily diversify their labor. They represented practically every production division of the factory, and although, their normal occupations had been limited to a highly developed system of mass production, they fitted into the garden plan admirably.

Those who enrolled in the garden force realized the importance of the opportunity provided them in the farming project and entered into the spirit of the work at hand as if they were compensated in money rather than in produce. Former employees, whose individual factory records were above the average, in the majority of cases also proved to be the best "farmers," substantiating the belief that mass-production methods and training can be applied successfully to an agricultural enterprise.

Costs.—The total average investment was about $50 per acre cultivated, as developed from the following itemized summary:

Item	Cost
Plowing, harrowing, planting, including labor, tractor rental, team hire, fuel, mechanical sewer, planter operation, and replacement parts	$1 900
Transportation of farm workers and materials, including use of trucks and passenger bus	1 550
Supervision (Farm Superintendent, part-time truck driver, and incidental specialized labor)	1 400
Spraying materials and equipment	1 100
Seeds, including seed potatoes	1 000
Fertilizer	800
Leases for land	750
Culverts, drain-tile, lumber for headquarters and distribution building, rental of storage trailers, road maintenance	250
Small equipment—tools	150
Incorporation and workmen's compensation	125
Miscellaneous	225
Total	$9 250
Interest, 6 months at 5%	231
Total investment	$9 481

$$\text{Investment per acre} = \frac{\$9\,481}{191.3 \text{ acres}} =, \text{ say, } \$50$$

Return on Investment.—The man who worked 1 day per week on the gardens during the season, without extra work or absence, received the items listed in Table 2, totaling in weight more than 1 ton. This quantity of vegetable food is sufficient to take care of a family of five for nine months. The total value of the produce was easily double the amount invested.

TABLE 2.—Garden Output per Man, Example 1

Produce	Unit	Quantity	Produce	Unit	Quantity	Produce	Unit	Quantity
Potatoes	Bushels	22	Summer squash	Pounds	31.5	Cayenne peppers	Dozen	16
Cabbage	Pounds	127	Rolled oats	Pounds	25.0	Sweet corn	Dozen	10
Tomatoes	Pounds	98	Greens	Pounds	23.0	Green onions	Dozen	7
Corn meal	Pounds	95	Pumpkins	Pounds	10.0	Hot peppers	Dozen	7
Turnips	Pounds	96	Peas	Pounds	8.0	Carrots	Dozen	4
Navy beans	Pounds	75	Kale	Pounds	5.0	Beets	Dozen	3
Green beans	Pounds	69	Dry onions	Pounds	1.0	Sweet peppers	Dozen	2

The efficiency of operation rapidly increased during the succeeding years. Although employment during 1934 had improved to such an extent that a continuance of the gardens was not necessary, the management felt that the experience was of sufficient value to continue with the work. The area cultivated was increased to 300 acres. The investment amounted to about \$20 000, or \$66 per acre. The value of the crop harvested and distributed, however, was \$60 000, or approximately double the amount per acre of the 1932 harvest.

Example 2

Another successful industrial gardening project, operated during 1932, 1933, and 1934 may be discussed as Example 2. About 90 acres of land were rented near the plant for gardening purposes and, in addition, employees were encouraged to cultivate individual gardens.

An agricultural expert was placed in charge of the entire project. Aside from the land and supervision, the Company furnished fertilizers, plowing and harrowing, all seeds and plants, and spraying and spray materials. Seeds and plants only were furnished to those with home gardens. The plant gardens were laid out in plots of 50 by 100 ft, with suitable spaces between lots for paths and driveways. This plan gave centralized operation, combined with the advantages incidental to the cultivation of individual plots. Planting schedules for each plot were planned with a view to providing good substantial food and a well-balanced diet.

Each plot holder was required to sign an agreement which set forth the obligations of both the Company and the gardener. The Company agreed to loan the plot, prepare the soil for planting, furnish the seed, and supervise the planting. The gardener agreed to cultivate the land and keep it free from weeds to the best of his ability. In return for this service the vegetables grown on the land became the sole property of the gardener, except in cases where willful negligence was evident. In this event, the Company reserved the right to revoke the agreement.

The employees on a part-time basis took a keen interest in the gardening work and readily grasped the opportunity to take part in the enterprise, once its advantages became evident. The number of plant gardens and home gardens for each year of operation were as follows:

Year	Plant gardens	Home gardens
1932	345	655
1933	587	813
1934	588	833

About 60% of the workers operating these gardens could be classed as skilled and the remainder as unskilled. Both classes were surprisingly proficient in gardening after the brief instruction period. As in the case of Example 1, the industrial employees who were most proficient in the plant obtained the best results with their gardens.

Accurate cost records were kept during 1933 and 1934 in order to determine the return on the investment. Actually, the gardeners, during these years, received vegetable produce equal in value to between five and

six times the total cash outlay. Aside from the monetary value of the product, other benefits were reflected in the improved health of the workers and the close relation that existed between the Company and the employees. The social value of this work in keeping idle hands and minds occupied and in supplying wholesome foods cannot be over-estimated.

Education as well as strict supervision is essential to the success of such a venture. In the beginning the Company had talks about gardening by a County farm adviser and agents of the State Department of Agriculture. Canning demonstrations and individual instruction in the care of produce for winter saving were given. Educational measures were directed toward developing a love of the work concurrently with gardening skill. Efforts in this direction were highly successful.

Summary of Experiences with the Diversification of Labor

The foregoing, together with the other experiments studied by the writers, lead to the following conclusions:

(1) The occupations of both common and skilled labor may be diversified successfully.

(2) Without exception, agriculture has been selected as the most satisfactory alternate occupation to industrial work.

(3) The most efficient factory employees generally obtain the best results from their garden plots.

(4) With a relatively high yield about 0.25 acre is necessary to provide a family of five with vegetable produce for the greatest part of a year. More land per worker is necessary if the soil is unsuited to intensive farming.

(5) The co-operative plan (that is, all the gardens centralized in one large plot) lends itself to the most efficient management. Mass-production methods may be applied to gardening as well as to manufacturing if this plan is adopted. The costs of plowing, harrowing, planting, and harvesting are reduced to a minimum if the gardening activities are conducted at one place. The principal disadvantage of operating one large plot on a labor-rotation basis is that workers may not have the pride or interest in their work that they would if they were responsible for the sole care of an individual garden. This may be overcome by adopting the plan of dividing the central plot into sections and assigning each section to individual workers (see Example 2).

The cultivation of small gardens adjacent to employees' homes is to be discouraged unless the workers or members of their families are skilled gardeners. This plan has produced only indifferent results as the costs of company operation and management are necessarily much higher than those for centralized plots.

(6) The service of a skilled agriculturist is absolutely essential. Few workers have the gardening experience that is required to insure them an adequate return on their efforts, therefore, additional educational measures are necessary. Lectures, educational bulletins, and planting schedules must be supplemented with individual instruction if the project is to be successful.

(7) Centralized storage facilities must be provided and arrangements made for distributing the produce.

The experiments outlined in the foregoing items do not indicate completely the possibilities of labor diversification. Most of these projects were emergency measures and had to be planned on a subsistence basis only. Regardless of this fact important lessons were learned and there now exists sufficient actual experience to state with certainty that labor diversification in connection with manufacturing operations could be both practical and profitable.

Tangible evidence has been presented to show the benefits that could be derived from both industrial decentralization and the diversification of labor. Nothing has been introduced, thus far, to show the improvements that could result from co-ordinating the operations of adjacent agricultural and industrial developments. The following two specific examples relating to irrigated areas show conclusively that economic conditions improved definitely after the introduction of industrial plants.

The Co-Ordination of Industry and Agriculture within Irrigation Districts

Yakima Valley, Washington.—Industries engaged in the processing of agricultural products followed irrigation development in the Yakima Valley. These industries consist of flour and feed mills, alfalfa mills, fruit dehydrator plants, fruit canning plants, meat packing plants, and various other types of plants for the packing, storing, handling, and distribution of crops. The canning plants alone employ approximately 3 000 annually for about 4 months and the dehydrating plants employ 800 to 1 000 for approximately 6 months.

The Federal Government has expended mostly in Yakima County or in irrigation projects approximately $33 000 000. As a result of these projects and their subsequent industrial development, the population of Yakima County increased from 13 462 in 1900 to 77 402 in 1930. During this same time the population of Yakima City increased from 3 154 to 22 101. The assessed valuation of the County has increased from $25 004 000 in 1900 to $233 041 000 in 1930, or almost tenfold.

Obviously, these benefits have been realized through agriculture, as a population base, being followed by factories for the processing of farm products.

Sugar Production in Sidney, Mont.—An outstanding case illustrating the improvement in the economic status of an irrigation district that follows the introduction of industrial plants may be found at Sidney, Mont. The Lower Yellowstone Irrigation Project was not successful as an agricultural venture alone. For a period of ten years the farmers of this District were faced with severe economic difficulties, with the result that water charges were defaulted.

In 1925 a sugar factory was constructed at Sidney which is centrally located in the Valley. At this time, few of the farmers understood the raising of sugar-beets and, in consequence, only about 50 000 tons of beets were produced annually within the District. There has been a gradual growth of

the industry, however, and, in 1935, the growers tributary to the Sidney factory produced approximately 200 000 tons, yielding a return to the farmers of approximately $1 500 000. As a result of linking the manufacture of beet sugar to the agricultural production of the Lower Yellowstone Irrigation Project the farmers of this District are now in good financial condition; many, in fact, are increasing their land holdings.

In the connection with the growth of sugar-beets the District also maintains a substantial live-stock development. During 1935, approximately 150 000 head (including lambs, sheep, and cattle) were fed for the market. This large feeding program is made possible by the by-products from both the factory and the farm. Beet tops and beet pulp and molasses make excellent foods in addition to hay.

The successful growing of sugar-beets calls for the rotation of crops on the farms. Alfalfa, the principal rotation crop, provides an ample quantity of hay for feeding purposes. Combining a sugar-beet and live-stock program has produced increased yields of all other crops grown in rotation with sugar-beets. Feeding of livestock is producing an abundance of manure which, through return to the soil, is gradually building up the fertility and yield of the farms.

PART II.—POTENTIAL INDUSTRIAL GROWTH OF CALIFORNIA RELATED TO CENTRAL VALLEY PROJECT

Irrigation as a Base for Western Industrial Development

Much of the land that will be made irrigable by the construction of hydraulic works in the Western States is adjacent to extensive markets for manufactured products. As these markets are now (1936) only partly supplied by local manufacturing establishments, opportunities exist for the profitable expansion of many industries in the Western States. This situation is especially favorable for decentralized industrial planning in relation to the land.

The benefits that would result from such an arrangement are obvious. Irrigation districts would benefit by having many of their products utilized by adjacent manufacturing plants. Hides, cotton, wool, cereals, sugar-beets, and fruits and vegetables are only a few of the agricultural products that could be processed by adjacent industries. In addition, these irrigation districts would benefit by having a substantial portion of their lands operated by industrial employees on a co-operative basis. This would tend to check the over-production of agricultural products on irrigated areas as the products of the co-operative gardens would be consumed principally by industrial employees. A third benefit would be the tremendous increase in population that could be supported by the land.

Industry would be likewise benefited by co-ordination with agriculture. The labor supply would be more stable and more efficient than in crowded urban locations. Industrial employees would have a "back-log" of agricultural employment and, therefore, would not face serious hardships in the event of a factory shut-down. An abundant supply of cheap power is available in most

irrigation districts. These advantages, together with nearness to markets and sources of raw material supplies should offer an incentive to many industries either to relocate their factories or to establish branch plants on projects that are adaptable to co-ordinated industrial and agricultural operations.

The large potential markets that now exist adjacent to recently developed irrigation areas would absorb most of the manufactured products at first. As the projects were settled additional markets would be developed in the immediate vicinity of the manufacturing plants. This would give opportunity for an industrial growth in proportion to the increase in population. This process could continue until the land had been fully developed.

Not only would agriculture and industry in the immediate vicinity of the irrigated areas be benefited by co-ordination, industry in all sections of the country would be helped by the extensive construction program and the additional manufacturing equipment that would be required by industrial decentralization.

The truth of the foregoing statements may best be illustrated by a detailed analysis of a selected irrigation project, the Madera Irrigation District in California, which is suitable for combined industrial and agricultural operations. It is impossible, however, to confine the investigation to the Madera District alone; it is first necessary to consider some of the industrial and economic aspects of both the State of California and the Central Valley Project.

For the purposes of this investigation the State of California was considered as a single-market area. An analysis was then made to determine the opportunities for industrial expansion that existed within the boundaries of this area in 1929. After this information was compiled, the capacity of the Central Valley Project to absorb a program of decentralized industrial development was studied broadly. A proportional part of the potential industrial growth for the entire State was then allocated to the district investigated. This consisted of a selected group of industries for which the location factors in the vicinity of Madera were favorable. It is not meant to imply that the Madera District offers the most favorable location in California for the industries selected. It was beyond the scope of this investigation to do more than to apply principles to a specific area. Similar analyses for other districts should indicate whether they too are suitable for combined industrial and agricultural development.

Population Growth Related to Land Development

The rate of industrial development has been found to bear a definite relation to population growth.[10] Population studies, therefore, are of primary importance in this investigation. In predicting California's future population sight must not be lost of the fact that a substantial part of the high rate of increase during past years has been due to immigration from other States and from abroad. More than 46% of the people now living in California came there during the past decade. The relationship between that part of the population born in California and the total for the State is shown by

[10] "Amended Application to Federal Emergency Administration of Public Works," p. 87.

Fig. 1(*a*), which also shows the forecast of population growth from 1932 to 1970, used by the Water Project Authority of the State of California as a basis for the growth curve of future increases in the area of irrigated lands Fig. 1(*b*).

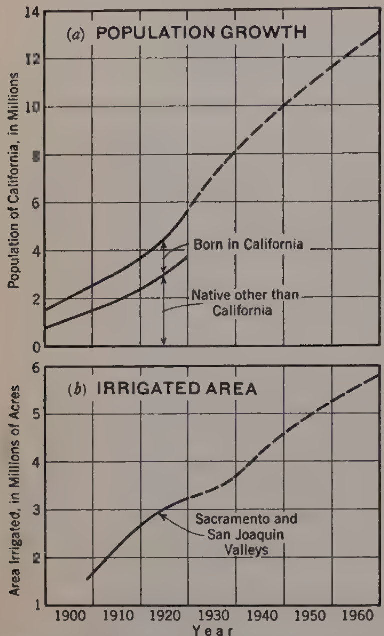


FIG. 1.—PAST AND ANTICIPATED GROWTH OF CALIFORNIA

AGRICULTURAL DEVELOPMENT

The total cropped area of the State is 8 390 000 acres.[11] Of this area, 4 746 000 acres, or about 57%, is irrigated. The irrigated lands of California, comprising 24% of the entire irrigated area of the United States, are among the most productive in the country. In 1929, California ranked eighteenth in the States of the Union in cropped area, second only to Texas in gross income from crop production. The per capita income of farm population, however, is greater for California than that for any other State.

INDUSTRIAL PATTERN OF CALIFORNIA

Industry has grown rapidly in California, responding to the westward movement of the center of population. From 1914 to 1929, the value of manufactured products increased 330% and the value added by manufacture, 410 per cent. In each case, this rate of increase was nearly twice that for the United States as a whole.

The 1929 industrial census for California does not present the distorted values that would be given by statistics compiled during the depression years. This investigation, therefore, presents both a cross-section of California industry in 1929 and a reconstruction of the opportunities for industrial expansion that existed in that year. As industrial development has not kept pace with population growth during the depression years there is, in 1936, an even greater opportunity for new industrial development in California than that indicated by the 1929 statistics.

The average value of industrial products per capita is nearly the same for the State of California as it is for the entire country. The following comparison of these values for the years 1929 and 1931 indicates the abnormal situation that existed in 1931 after the depression was well under way:

Description	1929	1931
Total Value of Industrial Products:		
For the entire United States	$70 435 000 000	$41 350 000 000
For California	3 103 000 000	1 985 000 000

[11] "Permissible Economic Rates of Irrigation Development in California," *Bulletin*, Univ. of California, p. 38.

Description	1929	1931
Value of Industrial Products per capita:		
For the entire United States	$577	$336
For California	565	330

A superficial examination of the foregoing statistics would seem to show that California was producing at least its proportionate share of the nation's manufactured goods and that little opportunity existed for profitable industrial expansion in 1929. A detailed study of California's manufactured products, however, shows that the picture is distorted greatly by several industries that serve a much larger population than that of the State. For example, in 1929, the industries in California engaged in the canning and preserving of fruits and vegetables served nearly 36 000 000 people on a proportionate basis; the motion picture industry served 85 500 000 people, and the petroleum refineries served 22 500 000 people.

The total value of the products of these three industries was $834 612 000 for the State of California and $3 574 000 000 for the entire United States. If these sums are deducted from the total value of all industrial products shown in the foregoing list it is found that the values of the remaining products per capita were $407 for California and $547 for the entire United States. These averages show conclusively that the majority of the industries in the State were not sufficiently developed to supply their adjacent markets.

The principal industrial areas in California are the Los Angeles District and the San Francisco-Oakland District. Fig. 2 shows the extent of each dis-

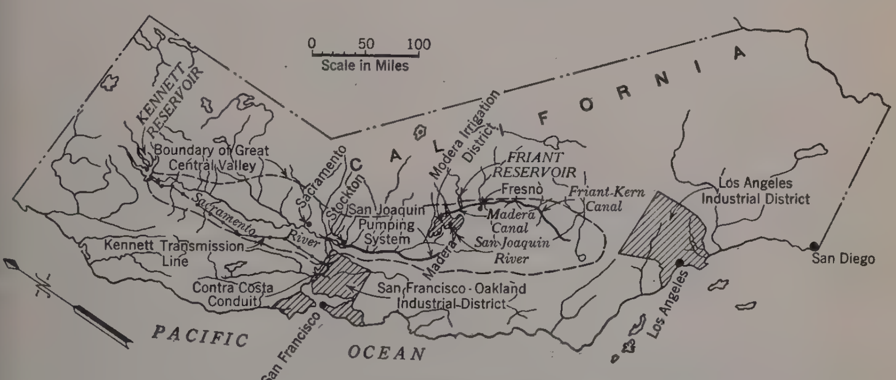


FIG. 2.—RELATION OF PRINCIPAL INDUSTRIAL DISTRICTS TO CENTRAL VALLEY AREA

trict and its relation to the Central Valley Area. Table 3 which gives the industrial characteristics of each of the principal districts, and of the remainder of the State, reveals that industry in California has followed the usual course; the greatest industrial concentration is found in the areas of greatest population density. This tendency of industry to follow its markets would not be altered by the adoption of an industrial decentralization program in connection with the Central Valley Project. Actually, decentralized planning would result in a broadening of this same policy. Many of the irrigated dis-

tricts of the Sacramento and San Joaquin Valleys are sufficiently close to the thickly populated areas to be satisfactory for both industrial and agricultural development. Furthermore, a co-ordinated industrial and agricultural program would enable these valleys to support a much greater population density than that which could derive a living from agriculture alone. The distribution of industries throughout the Central Valley therefore, would have a centripetal effect on their respective market areas. Not only would these industries be situated favorably, as far as the existing potential markets are concerned, they would also have the ultimate result of creating important new markets which would develop inwardly toward the factories as focal points.

TABLE 3.—CHARACTERISTICS OF PRINCIPAL INDUSTRIAL DISTRICTS IN CALIFORNIA

Industrial district	Population in 1930	Area, in square miles	Population per square mile, in 1930	Persons engaged in industry in 1929	Total value of products in 1930
Los Angeles	2 209 000	4 115	537	146 000	$1 319 000 000
San Francisco–Oakland	1 307 000	2 464	530	114 000	1 166 000 000
Remainder of State	2 161 000	149 073	14.5	102 000	618 000 000
Total	5 677 000	155 652	36.4	362 000	$3 103 000 000

Industrial Survey for California

In order to get a true concept of the deficiency of capacity of the California manufacturing concerns in 1929, it is necessary to make a detailed study of each of the 327 industries classified by the industrial census. If the population served by each industry in the State is computed from the census statistics it is possible to establish for each product the potential markets that exist within the State boundaries.

The most satisfactory method of estimating the population served by each industry is based on the average values of manufactured products per capita; for example, the total value of the products in 1929 of the boot and shoe industry was about $966 000 000 for the entire United States; the average value of product per capita, based on a population of 122 000 000, was $7.92. For the same year the total value of the products of the boot and shoe industry in California was $3 095 000. Dividing this sum by the average value of $7.92 per person gives a total population of 391 000 served by the California factories. The difference between the population of the State (5 500 000 in 1929) and the population served by the industry is 5 109 000 persons who purchased shoes that were manufactured outside the State. Further analysis, including estimates of the population per establishment and the employees per establishment, indicates that 56 additional average shoe-manufacturing establishments, employing a total of 9 400 persons, could have been established in California in 1929. Table 4[12] shows the results of an analysis, similar to the foregoing, for each industry that could have been expanded in California in 1929.

[12] Computed from statistics in United States Industrial Census, 1929.

TABLE 4.—Detailed Study of California Industries

Item	United States, population per plant	Present number of plants in California	Potential additional plants in California	Potential additional number of industrial employees in California	Potential additional value of industrial products in California
Total for all industries	....	......	1 742	164 273	$940 201 000
Agricultural implements	415 000	18	5	815	$4 600 000
Aluminum manufacturers	812 000	14	6	920	5 820 000
Ammunition and related products	5 800 000	......	1	379	1 965 000
Artificial leather	7 170 000	......	1	142	1 500 000
Artists' materials	1 790 000	4	3	62	383 000
Asbestos products other than steam packing and pipe and boiler covering	1 820 000	......	3	420	2 520 000
Asphalted felt base floor covering	8 700 000	......	1	124	1 440 000
Bags, paper, exclusive of those made in paper mills	1 285 000	......	4	357	2 920 000
Baking powders, yeast, and other leavening compounds	2 560 000	......	2	164	2 370 000
Baskets and rattan and willow ware, not including furniture	447 000	......	12	460	1 030 000
Belting, leather	590 000	10	4	74	745 000
Blacking stains and dressings	717 000	12	4	61	620 000
Blueing	6 100 000	......	1	7	62 000
Bolts, nuts, washers, and rivets not made in plants operated in connection with rolling mills	1 045 000	5	4	580	3 270 000
Bone black, carbon black, and lamp black	1 570 000	......	4	97	910 000
Boot and shoe cut stock not made in boot and shoe factories	580 000	......	10	480	6 050 000
Boot and shoe findings not made in boot and shoe factories	320 000	......	17	560	2 640 000
Boots and shoes, other than rubber	90 700	12	56	9 400	41 000 000
Boxes, paper, not elsewhere classified	98 000	38	13	670	3 040 000
Brooms	300 000	19	4	65	180 000
Brushes other than rubber	400 000	8	9	270	1 340 000
Buttons	508 000	......	11	460	1 305 000
Candles	6 100 000	......	1	47	300 000
Carbon paper and inked ribbons	2 140 000	4	2	62	595 000
Card cutting and designing	1 740 000	......	3	130	887 000
Carpets and rugs — wool, other than rag	1 820 000	......	3	1 569	8 000 000
Carriages, children's	1 540 000	......	4	360	1 305 000
Cars, electric and steam railroad, not built in railroad repair shops	828 000	......	7	2 100	14 800 000
Cash registers and adding, calculating and card tabulating machines	2 950 000	......	2	820	4 950 000
Cast-iron pipe	1 700 000	......	3	942	3 610 000
Cereal preparations	1 010 000	9	4	288	6 100 000
Cheese	44 300	37	21	43	862 000
Chemicals, not elsewhere classified	221 000	48	7	900	8 960 000
Chewing gum	3 300 000	......	2	130	2 710 000
Cleaning and polishing preparations	285 000	34	11	120	1 320 000
Clocks, clock movements, time recording devices, and time stamps	2 170 000	......	3	550	1 900 000
Clothing (except work), men's, youths' and boys' (not elsewhere classified)	33 000	88	130	6 000	31 600 000
Clothing, women's, not elsewhere classified	15 100	305	182	5 000	38 600 000
Combs and hairpins not made from metal or rubber	5 100 000	......	1	32	112 000
Corn syrup, corn sugar, corn oil and starch	3 500 000	......	2	370	7 500 000
Corsets and allied garments	572 000	......	10	720	3 470 000
Cotton goods	95 000	8	53	18 000	62 700 000
Cotton, small wares	605 000	3	9	730	2 840 000
Cutlery (not including silver and plated ware), and edge tools	510 000	7	11	755	3 440 000
Dairymen's supplies, creamery, cheese factory, and butter factory, equipment; poultrymen's and apiarists' supplies	710 000	15	5	214	1 250 000
Dyeing and finishing textiles	167 000	12	30	3 690	19 300 000
Electrical machinery, apparatus, and supplies	67 500	138	48	10 750	61 400 000
Embroideries	150 000	14	27	270	830 000
Emery wheels and other abrasive and polishing appliances	1 800 000	......	3	227	1 415 000
Engraving (other than steel, copper-plate, or wood), chasing, etching, and die sinking	650 000	11	4	56	215 000
Engraving, steel and copper-plate, and plate printing	267 000	4	8	224	850 000
Envelopes	712 000	9	1	73	352 000
Excelsior	1 850 000	......	3	63	225 000
Explosives	1 280 000	......	4	294	3 260 000
Fancy and miscellaneous articles not elsewhere classified	125 000	55	13	350	1 150 000

TABLE 4.—(*Continued*)

Item	United States, population per plant	Pesent number of plants in California	Potential additional plants in California	Potential additional number of industrial employees in California	Potential additional value of industrial products in California
Felt goods, wool, hair, or jute	2 230 000	......	3	285	$2 050 000
Fertilizers	191 000	21	14	550	4 950 000
Files	3 700 000	......	2	194	640 000
Firearms	5 800 000	......	1	350	1 000 000
Fire extinguishers, chemical	4 350 000	......	1	50	385 000
Fireworks	2 440 000	......	2	90	297 000
Flour and other grain - mill products	30 400	54	57	550	14 900 000
Forgings, iron and steel, not made in plants operated in connection with rolling mills	557 000	8	6	635	4 170 000
Foundry supplies	2 080 000	......	3	52	472 000
Furnishing goods, men's, not elsewhere classified	204 000	15	9	450	2 300 000
Gas machines, gas meters, and water and other liquid meters	1 690 000	6	1	105	523 000
Glass	464 000	6	3	920	3 790 000
Glass products (except mirrors) made from purchased glass	283 000	25	12	200	900 000
Gloves and mittens, leather	473 000	17	5	190	720 000
Glue and gelatine	1 645 000	5	2	70	650 000
Gold-leaf and foil	1 625 000	6	2	40	115 000
Handkerchiefs	....	......	5	294	1 330 000
Hardware, not elsewhere classified	252 000	27	18	2 200	8 200 000
Hat and cap materials, men's	1 090 000	......	5	133	1 135 000
Hats and caps, except felt and straw, men's	212 000	32	6	78	383 000
Hats, fur-felt	767 000	6	6	720	3 960 000
Hats, straw, men's	2 440 000	......	2	172	840 000
House-furnishing goods not elsewhere classified	127 000	89	8	166	1 110 000
Ink, printing	890 000	6	2	36	465 000
Ink, writing	4 200 000	......	1	35	207 000
Instruments, professional and scientific	442 000	14	12	885	3 750 000
Iron and steel — steel works and rolling mills	251 000	19	16	14 500	113 000 000
Jewelry	79 200	111	33	732	3 720 000
Jewelry and instrument cases	1 080 000	4	4	94	318 000
Jute goods	6 100 000	......	1	267	987 000
Knit goods	64 500	48	64	7 500	30 200 000
Lace goods	2 900 000	......	2	348	1 320 000
Lasts and related products	2 100 000	......	3	97	346 000
Leather goods, not elsewhere classified	310 000	26	9	163	810 000
Leather, tanned, curried, and finished	258 000	16	17	1 950	17 300 000
Lubricating oils and greases not made in refineries	687 000	13	6	128	2 160 000
Machine tool accessories and small metal-working tools not elsewhere classified	169 000	25	27	1 185	5 480 000
Machine tools	435 000	......	13	2 520	11 100 000
Motor vehicle bodies and motor vehicle parts	106 000	94	39	8 200	52 500 000
Motorcycles, bicycles, and parts	5 800 000	......	1	242	1 070 000
Musical instrument parts and materials, piano and organ	1 820 000	9	2	79	260 000
Musical instruments and parts and materials not elsewhere classified	1 150 000	......	5	190	650 000
Nails, spikes, etc., not made in wire mills, or in plants operated in connection with rolling mills	2 210 000	......	3	110	578 000
Needles, pins, hooks and eyes, and snap fasteners	2 840 000	......	2	316	1 000 000
Oil, cake, and meal, cottonseed	220 000	8	15	530	8 320 000
Paper	177 500	12	19	3 200	26 500 00
Pulp (wood and other fiber)	615 000	......	9	1 220	10 750 000
Paper goods, not elsewhere classified	280 000	14	13	720	5 300 000
Patent or proprietary medicines and compounds	80 000	117	20	340	4 200 000
Paving materials: Asphalt, tar, crushed slag, and mixtures	965 000	5	3	52	552 000
Pencils, lead, including mechanical	320 000	......	2	314	1 260 000
Pens, fountain and stylographic; pen points, gold, steel and brass	1 600 000	......	3	244	1 550 000
Perfumes, cosmetics and other toilet preparations	149 000	55	24	576	5 650 000
Pocketbooks, purses, and card cases	420 000	11	11	451	2 590 000
Printing materials not including type or ink	1 450 000	4	3	38	247 000
Pulp goods	2 710 000	......	2	175	1 230 000
Rayon and allied products	4 200 000	......	2	1 870	6 780 000
Refrigerators, mechanical	3 580 000	......	2	872	7 300 000
Regalia, badges and emblems	1 580 000	......	4	98	358 000
Rubber goods other than tires, inner tubes, and boots and shoes	295 000	28	6	678	3 560 000
Saddlery and harness	468 000	7	7	107	587 000
Safes and vaults	4 500 000	......	1	171	875 000

TABLE 4.—(*Continued*)

Item	United States, population per plant	Present number of plants in California	Potential additional plants in California	Potential additional number of industrial employees in California	Potential additional value of industrial products in California
Saws	1 510 000	......	4	230	1 025 000
Scales and balances	2 070 000	......	3	235	1 375 000
Screw-machine products and wood screws	447 000	9	11	945	4 450 000
Sewing machines and attachments	3 130 000	......	2	550	2 040 000
Shirts	141 000	18	26	1 920	6 850 000
Shortenings, not including lard, and vegetable cooking oils	3 050 000	......	2	158	6 980 000
Silk and rayon manufactures	81 500	......	67	6 250	33 000 000
Silversmithing and silverware	1 465 000	6	3	240	1 180 000
Soda-water apparatus	2 250 000	3	2	134	907 000
Sporting and athletic goods not including firearms and ammunition	503 000	12	10	527	2 340 000
Springs, steel, except wire, not including plants operated in connection with rolling mills	1 415 000	8	2	165	1 160 000
Stamped ware, enamel ware, and metal stamping, enameling, japanning and lacquering	210 000	23	17	1 370	6 100 000
Stationery goods not elsewhere classified	600 000	9	7	462	2 520 000
Steam and other packing pipe and boiler covering and gaskets, not elsewhere classified	712 000	13	2	115	595 000
Stereotyping and electrotyping not done in printing establishments	530 000	7	5	1 850	830,000
Structural and ornamental iron and steel work not made in rolling mills	82 000	136	24	1 120	7 700 000
Surgical and orthopedic appliances	336 000	29	12	350	2 350 000
Suspenders, garters, elastic goods made from purchased webbing	1 340 000	......	4	230	1 370 000
Tanning materials, natural dyestuffs, mordants and assistants, and sizes	967 000	......	6	143	1 800 000
Textile machinery and parts	324 000	......	17	1 425	5 500 000
Tin cans and other tinware not elsewhere classified	525 000	......	11	1 600	13 350 000
Tools, not including edge tools, machine tools, files, or saws	228 000	20	19	830	3 500 000
Toys (not including wagons and sleds), games and playground equipment	256 000	16	17	710	2 700 000
Trimmings (not made in textile mills) and stamped art for embroidery	191 000	24	19	292	1 350 000
Typewriters and parts	......	......	1	832	2 790 000
Umbrellas, parasols, and canes	863 000	......	6	156	902 000
Washing machines, wringers, dryers, and ironing machines for household use	1 880 000	5	3	410	3 400 000
Waste	895 000	6	4	140	1 100 000
Wire drawn from purchased bars or rods	1 500 000	......	4	1 140	9 600 000
Wood, turned and shaped, and other wooden goods not elsewhere classified	137 000	32	25	610	1 970 000
Woolen goods	265 000	5	18	2 500	11 600 000
Wool pulling	6 770 000	......	1	38	612 000
Wool scouring	5 800 000	3	1	71	94 000
Wool shoddy	2 440 000	......	2	82	693 000
Worsted goods	462 000	......	12	4 270	24 300 000

A summation of the deficiencies computed in Table 4 shows that in 1929 the value of products manufactured within the boundaries of California could have been increased by about one-third, or approximately $1 000 000 000, if the needs of the population within the State had been served by local manufacturing establishments. If California industries had taken advantage of this opportunity 1 742 new plants, employing more than 164 000 persons could have been established. Perhaps a clearer way of visualizing these deficiencies is to state that an additional industrial development, nearly equivalent in size to that in either Los Angeles County or in the San Francisco-Oakland District, could have found profitable markets within the State of California in 1929.

The Central Valley Project

The industrial deficiencies summarized in Table 4 are related intimately to the inadequate water supplies of the southern part of the State. This condition will be corrected by the Central Valley Project.

The principal construction features of this project (see Fig. 2) are the Kennett Dam and power plant, the Kennett transmission line and sub-station, the Keswick Afterbay Dam and Reservoir, the Keswick power plant, the Sacramento-San Joaquin Delta cross-channel, the Contra Costa conduit, the Friant Dam and power plant, the Madera Canal, the Friant Kern Canal, and the San Joaquin pumping system.

Location Factors

In order to select a suitable location for each of the industries listed in Table 4, consideration must be given to the following factors: (1) Water supply; (2) proximity to supply and quality of raw materials; (3) location of markets for the finished product; (4) transportation costs; (5) State and local taxes; (6) power costs—hydro-electric, oil, and coal; (7) sources of labor supply, labor rates and attitude of those in control of State and local politics in controversies between labor and capital; (8) living costs; (9) climatic conditions and general health; (10) social conditions; and, (11) availability of industrial sites.

The relative weight of each of the foregoing factors will vary greatly with the particular industry; for instance, industries employing large numbers of men are certain to rate labor factors high. Plants producing heavy or bulky products will place special emphasis on transportation costs and proximity to markets. The development of such industries within certain sections of the Central Valley will be made economically possible by the low-cost water transportation that will follow the channel improvement of the San Joaquin and Sacramento Rivers. Water supply is a particularly important factor in determining the location of breweries, creameries, and packing plants.

It cannot be overlooked that many industries have not followed closely either cheap power and raw materials or favorable living conditions; for example, iron and steel mills are not located in Northern Minnesota and many textile mills are not near cotton or wool-producing sections. Markets, labor supply, and transportation facilities are generally given greater weight than the other factors in locating all industries except those producing foodstuffs. As a typical example, the iron and steel industry has been satisfactorily located on the southern shore of Lake Michigan, because at that place iron ore from Duluth, Minn., and cheap coal from Southern Illinois meet at a point adjacent to the markets and an adequate dependable labor supply.

It would not be possible, of course, to expand all the industries listed in Table 4 within a short period of time. New ones could be developed concurrently with the irrigated lands that will be made available by the Central Valley Project. These lands will form an agricultural base which will be capable of supporting a rapid population growth and, if industrial expansion

is correlated with agricultural growth, the population density will increase at even a greater rate.

Correlation of Industry and Agriculture

For example, assume that much of the industrial deficiency shown by Table 4 will be made up in the period between 1935 and 1945 through a co-ordinated industrial and agricultural development. Fig. 1(*b*) indicates that the increase in the area of irrigated lands of the State during that period will be about 800 000 acres and Fig. 1(*a*) that the population increase for the State will be about 2 000 000.

It is reasonable to assume that between 1935 and 1945 200 000 industrial employees, in addition to those employed in California in 1929, can be settled permanently in rural districts and can divide their working time between agriculture and industry. If 1 acre of land were allotted to each employee for gardening purposes, only one-fourth the increase in the area of irrigated land would be devoted to co-ordinated agricultural and manufacturing operations.

To provide for the continued future development of irrigated lands, as indicated by Fig. 1(*b*), more water will be required than that which will be furnished by the present Central Valley construction program. These additional supplies will be secured from other reservoirs contemplated in the State's comprehensive plan of water development.

The foregoing examples indicate broadly the opportunities for industrial expansion that exist within the State and also show that future industrial growth could be absorbed by the Central Valley without changing materially the program of agricultural development now planned for California. Although this exposition demonstrates the relation of the entire Central Valley to potential industrial development, it does not show in detail the economic benefits that could be obtained by co-ordinating agriculture and industry. A more comprehensive picture of the possibilities of decentralized industrial planning is needed, and this may best be obtained by a detailed study of a specific project.

PART III.—POSSIBILITIES OF MADERA IRRIGATION DISTRICT AS LOCATION FOR CO-ORDINATED INDUSTRIAL AND AGRICULTURAL PROJECTS

General Description

The Madera Irrigation District in California is suitable for combined industrial and agricultural development owing to its proximity to important markets, its climate, its large undeveloped areas of fertile soils, and its available labor supply. Fig. 2 shows its relative location. This District is situated in the heart of the San Joaquin Valley, about 150 miles southeast of San Francisco, 150 miles southwest of Stockton, and 22 miles northwest of Fresno, the latter being the largest city in the San Joaquin Valley. The San Joaquin River comprises the southern boundary of the District and the Chowchilla River, the northern boundary. The San Joaquin River has an annual run-off of about 1 750 000 acre-ft per yr, most of which comes down

in the flood season of May and June. The Chowchilla River is a flood-water stream, with an average run-off of about 50 000 acre-ft per yr. The District is intersected about midway by the Fresno River, which is also a flood-water stream, with an average run-off practically equal to that of the Chowchilla River.

The total area of the District is 173 179 acres, of which 32 719 acres are rough hardpan, of doubtful value for irrigated land under present economic conditions, and 13 872 acres are irrigable, but not as satisfactory for cultivation as the better class of land. The remainder of the District, or 126 588 acres, is first-class, flat, irrigable land, of which 75 000 acres are now irrigated and cultivated and 51 588 acres are not irrigated. The latter area, which is now (1936) partly used for raising grain, is available for either industrial co-operative gardening or for independent agricultural development. For the purpose of this investigation, it has been assumed that about one-half the first-class available land, or 25 000 acres, will be taken up by combined industrial and agricultural developments.

Population and Labor Supply

The Madera Irrigation District includes two cities within its limits, namely, Madera, with a population of about 6 000, and Chowchilla, with a population of 1 000. The rural population is estimated to be about 7 000, making a total population for the District of 14 000. Aside from these permanent residents, there is a large transient laboring element which follows the cotton and fruit seasons.

Most of the labor available is unskilled. If permanent employment were offered, a large portion of the transient labor would settle permanently in the District and skilled labor would also be attracted. A number of workers are available in winter when they are not engaged in agriculture, and many men who work in the granite quarry and logging camps have extensive "off" seasons. As living conditions within the District are excellent, no difficulties would be experienced in attracting an ample supply of both skilled and unskilled labor.

Climate and Rainfall

The average daily maximum temperature for the year is 75 degrees. The temperature varies from a minimum of 36° in January to a maximum of 99° in July. The climate is dry and exceptionally healthful. Regardless of the high maximum temperatures that are attained during the summer, personal discomfort is seldom experienced owing to the low humidity.

The average annual rainfall is 9.56 in., which is typical for the arid lands of the San Joaquin Valley requiring irrigation. About 85% of the seasonal rainfall comes in the six months between November and April, with a yearly average of about 45 days of rain. There is a peculiar tendency, especially in the spring and autumn, for the rain to occur at night; so there is little interference with work. The climatic conditions of the San Joaquin Valley are especially favorable to industrial development.

Water Requirements and Water Supply

For the purposes of this investigation an analysis has been prepared on the premise that the District was fully developed in 1929. In the following analysis, it has been assumed that 25 000 acres of the first-class land has been utilized by industry for co-operative gardening purposes and that the remainder is developed by individual farmers.

Probably not more than 85% of the available land, or 119 000 acres, will be irrigated in any single year. The remaining 15% may be dry farmed, temporarily, or allowed to lie fallow. The quantity of water that will be required by the project, therefore, will depend upon the relative acreages of the respective crops that are grown upon this land. Table 5 shows the land use assumed in making this investigation and also the water requirements for each crop.

TABLE 5.—Land Use Assumed in Investigation

Crop	Area, in acres	Duty, in acre-feet per acre	Total consumption, in acre-feet
Alfalfa	28 200*	3.0	84 600
Cotton	18 800*	2.5	47 000
Trees and vines	23 500*	2.0	47 000
Grain	9 400*	1.5	14 100
Miscellaneous field crops	9 400*	3.0	28 200
Truck	4 700*	3.0	14 100
Industrial gardens	25 000	3.0	75 000
Total for irrigation	119 000	2.6	310 000
Domestic and industrial use based on future population of 150 000	....	....	17 000
Total annual consumption	....	....	327 000

* Based on percentages used in forecasting demands for Central Valley Project.

A large part of this required supply may be drawn from the San Joaquin and Fresno Rivers after the Central Valley Projects have been constructed. Experience in this District has shown that 25% of the gravity water diverted from the San Joaquin River will be lost by lateral seepage and of this quantity 15% will be recovered by pumping, the other 10% being completely lost. The quantity of water, in acre-feet, to be taken from each source of supply, therefore, is estimated as follows:

From Fresno River	20 000
From San Joaquin River, 258 000 acre-ft, of which three-fourths reaches the land	193 000
Pumped from natural underground recharge	75 000
Pumped from artificial recharge, 15% of 258 000 acre-ft	39 000
Total delivered to land and industries	327 000

This quantity is only about 10% in excess of that which would be required if the District were developed as an agricultural project alone.

Cost of Water Supply

It has been estimated that the cost of water from the San Joaquin and Fresno Rivers will be $3.00 per acre-ft after the construction of the Central

Valley Projects. The average cost of pumping in the Madera District is $0.05 per acre-ft for each foot of lift. For the average lift of 65 ft which exists at present, the average total cost of pumping is $3.25 per acre-ft. This value will apply only after the Central Valley projects are constructed. As the ground-water is now (1936) lowering rapidly, pumping charges will be increased if new sources of supply are not obtained. On the basis of the foregoing unit costs, the total cost of water may be estimated as follows:

Gravity supply, 278 000 acre-ft @ $3.00......	=	$834 000
Pumped supply, 114 000 acre-ft @ 3.25......	=	370 000
Operation and maintenance of gravity system	=	120 000
Total cost of water per year...........	=	$1 324 000

$$\frac{\$1\,324\,000}{327\,000 \text{ acre-ft}} = \$4.05 \text{ per acre-ft, or only } 1\tfrac{1}{4} \text{ cents per } 1\,000 \text{ gal, at the land or factory}$$

The proportion of the total annual cost required for each use would be:

Industrial gardens,	75 000 acre-ft @ $4.05	=	$303 750
Industrial and domestic use,	17 000 acre-ft @ 4.05	=	68 500
Independent agricultural development,	235 000 acre-ft @ 4.05	=	951 750
Total	327 000 acre-ft	=	$1 324 000

The construction of the Central Valley Projects will make available an adequate and low-cost supply of water for both agricultural and industrial purposes.

Power

An ample quantity of power is available for both industrial and agricultural needs. Power charges vary from a minimum of 55 cents per kw-hr for a monthly consumption of more than 5 000 kw-hr, to a maximum of 4.0 cents per kw-hr for general power service.

Soils

The soils of the Madera District consist principally of sandy loams of excellent fertility, suitable for raising alfalfa, grains, trees and vines, truck, and other field crops.

Land Ownership and Value

All the land in the District is privately owned although a large part of it is in the hands of banks and mortgage companies. This situation, which is not uncommon to irrigation districts, as well as other farming areas, points to the need for a re-adjustment in irrigation planning. Obviously, a greater population density is needed, and this may be obtained most readily by industrial developments related to the land as a population base.

Holdings in the District range in size from about 20 acres to 320 acres, with an average of about 80 acres. Values of agricultural land depend at present upon the relative fertility of the soil and the comparative ease with

which underground water may be developed. Grain lands, considered non-irrigable under present conditions, are worth from $18 to $25 per acre. Land adapted to the growing of cotton, vines, deciduous fruit, etc., is valued at from $75 to $125 per acre.

Transportation

The Madera District is traversed by the Southern Pacific and the Atchison, Topeka, and Santa Fé Railroads which connect it with the most important market areas. Many of the products of the District are transported by truck to Fresno where they are taken over by jobbers and packers for refrigeration and shipment. The average length of haul to Fresno is about 25 miles, and the hauling rates vary from 8 to 12 cents per ton-mile.

Value of Crops

An estimate of the annual gross value of the crops, if the District were completely developed, is given in Table 6. The prices upon which this estimate is based are the average f.o.b. values per acre for the seasons of 1929 to 1934, inclusive.

TABLE 6.—Value of Crops

Crop	Area, in acres	Value per acre	Total gross value
Industrial gardens	25 000	$100,00	$2 500 000
Alfalfa	28 200	66,00	1 870 000
Cotton	18 800	55,00	1 040 000
Trees and vines	23 500	94,00*	2 210 000
Grain	9 400	19,00	180 000
Miscellaneous field crops	9 400	50,00	470 000
Truck	4 700	100,00	470 000
Total	119 000	$73.44	$8 740 000

* Based on average value for San Joaquin Valley; derived from data given in "Permissible Economic Rate of Irrigation Development."

The average gross value of crops per acre of crops for the years 1929 to 1934 is about two-thirds of the value for 1929[13]. On this basis the average gross value per acre for the Madera District in 1929 would have beeen $110 per acre, and the total gross value of crops for the District would have been $13 090 000.

The gross value per acre of crops for the Madera District, on the 1929 basis, compared favorably with those of the most productive Federal Reclamation Projects, as follows:

Project	Gross average value of crops per acre (1929 basis)[14]
Okanogan, Washington	$250
Yakima, Washington	120
Madera, California (estimated)	110
Yuma, Arizona	90
Carlsbad, New Mexico	75

[13] Rept. on Federal Reclamation, by J. W. Haw and F. E. Schmitt, M. Am. Soc. C. E., p. 63.

[14] *Loc. cit.*, pp. 34 to 58, inclusive.

Existing Industrial Plants

There are seven cotton gins scattered throughout the Madera District from Chowchilla to a region eight miles south of Madera. There is also a cotton-seed mill and a creamery at Chowchilla. There are three wineries within the project, with a combined tankage of 2 757 000 gal, and also a lumber mill with a capacity of 5 000 000 fbm per season.

Future Industrial Development

It is impossible, of course, to forecast accurately the kinds of industries that may be attracted by a given situation. All that may be accomplished in this investigation is to demonstrate the possibilities of co-ordinating industry and agriculture within the District by studying the characteristics of the area and of a selected group of manufacturing plants. Table 7 gives a list of industries that could be located within the Madera District and also certain average relationships which have beeen computed from the statistics for these industries in the Industrial Census of 1929.

TABLE 7.—Industries That Could Be Co-Ordinated with Agriculture on Madera Irrigation District

Industries	Persons in industry	Number of establishments	Salaries	Wages	Cost of materials	Value of products	Value added by manufacture
(*a*) Industries Engaged in Processing Agricultural Products							
Leather, tanned, curried and finished..........	2 000	17	$576 000	$2 340 000	$12 400 000	$17 700 000	$5 300 000
Boots and shoes........	5 000	30	1 000 000	4 950 000	11 400 000	21 400 000	10 000 000
Pocketbooks, purses, and card cases...........	500	11	193 000	635 000	1 470 000	2 840 000	1 370 000
Cotton goods..........	5 000	15	470 000	3 720 000	10 400 000	17 400 000	7 000 000
Woolen goods..........	1 000	7	222 000	1 060 000	2 650 000	4 650 000	2 000 000
Knit goods............	3 200	27	615 000	3 060 000	6 500 000	12 900 000	6 400 000
Wood, turned and shaped, other wooden goods...	500	20	130 000	425 000	660 000	1 620 000	960 000
Sub-Total.........	17 200	127	$3 206 000	$16 190 000	$45 480 000	$78 510 000	$33 030 000
(*b*) Industries Engaged in Manufacturing Mechanical and Electrical Equipment.							
Agricultural implements.	800	5	$225 000	$990 000	$1 910 000	$4 600 000	$2 690 000
Electrical machinery....	5 000	22	2 110 000	5 650 000	12 000 000	28 500 000	16 500 000
Refrigerators, mechanical	1 000	2	362 000	1 510 000	3 740 000	8 350 000	4 610 000
Hardware manufacturing	1 000	8	328 000	1 120 000	1 270 000	3 810 000	2 540 000
Sub-Total.........	7 800	37	$3 025 000	$9 270 000	$18 920 000	$45 260 000	$26 340 000
Totals............	25 000	164	$6 231 000	$25 460 000	$64 400 000	$123 770 000	$59 370 000

In listing this group, it was assumed that about one-eighth the total industrial deficiency for the State (see Table 4) would be developed within the Madera District. To have concentrated more industries in this area (on the 1929 basis) would have defeated the purposes of an industrial decentralization program. The plants were selected, therefore, with a view to obtaining a well diversified group producing a total annual value of product equal in amount to about $124 000 000.

The selection of the industries shown by Table 7 was based on the following principles:

(1) Industries were selected that would not compete with existing industries within the Madera District.

(2) The location factors, outlined previously under the heading "Possibilities of the Madera Irrigation District as a Location for Co-Ordinate Industrial and Agricultural Projects", are favorable to each of these industries.

(3) All manufacturing plants listed in Table 7 could have their operations co-ordinated readily with agricultural work, that is, workers could divide their time between the shop and their gardens without interfering with industrial operations.

(4) Two groups of industries were selected: The first relates to the processing of agricultural products, and the second consists of certain industries using mass-production methods in their manufacturing processes. Both groups are compatible with a program of decentralization. Industries engaged in the processing of agricultural products that could be raised within the District would effect large savings in transportation costs and offer ready markets for the raw materials produced by adjacent farms. Workers engaged in mass-production operations, as previously emphasized in this paper, need greatly both the physical diversification and the "back-log" of employment that may be obtained through part-time gardening.

(5) The annual value of the products of each of these plants is either equal to or less than the deficiency of this industry within California in 1929.

(6) Raw materials for each industry may either be produced within the District or brought in by rail or water transportation from adjacent sources of supply.

Approximately 25 000 industrial workers would be required to operate the plants shown by Table 7. As each of these would represent a family of about four, it is reasonable to assume that the total increase in population due to this industrial development would be 100 000. Allowing 1 acre of land per worker for gardening purposes, a total of 25 000 acres would be required for decentralized industrial developments as assumed previously herein.

Markets and Competition

The existing plants in California, manufacturing the products listed in Table 7, would offer little competition to similar industries located in the vicinity of Madera. This fact is shown clearly by Table 8 which gives an analysis of the existing market served by each industry shown in Table 7.

The three industrial areas analyzed in Table 8 are outlined in Fig. 2. Values of products and the population served for each industry were segregated for these three Districts. It will be observed from Table 8 that the population served by most of the industries within each District is now appreciably less than the population of the District. This means that the existing industries are not even meeting the needs of their adjacent areas and, therefore, would not offer much competition outside these areas.

Consider, for example, the products of the plants manufacturing boots and shoes. In the Los Angeles industrial area, the shoe factories served only 203 000 persons, leaving a market of more than 2 000 00 persons in this area who must purchase shoes that are manufactured elsewhere. Table 8 shows

TABLE 8.—ANALYSIS OF MARKETS AND COMPETITION

Industry	State of California		Los Angeles Industrial Area		San Francisco Oakland Industrial Area		Remainder of State	
	Value of product, in thousands of dollars	Population served by industry, in thousands	Value of product, in thousands of dollars	Population served by industry, in thousands	Value of product, in thousands of dollars	Population served by industry, in thousands	Value of product, in thousands of dollars	Population served by industry, in thousands
Population,* 1930 census, in thousands	......	5 677	......	2 209	......	1 307	......	2 161
Leather, tanned, curried and finished	4 410	1 120	....	....	2 946	750	1 464	370
Boots and shoes	3 095	391	1 604	203	....	....	1 491	188
Pocketbooks, purses, and card cases	492	882	298	532	194	350	....	....
Cotton goods	6 406	513	....	....	....	....	6 406	513
Woolen goods	1 561	655	....	....	....	....	1 561	655
Knit goods	10 303	1 400	6 341	862	....	....	3 962	538
Wood, turned and shaped; other wooden goods	1 159	2 040	645	1 130	410	725	104	185
Agricultural implements	7 924	3 480	....	....	....	....	7 924	3 480
Electrical machinery	42 131	2 240	11 009	588	26 865	1 425	4 257	227
Refrigerators, mechanical	....	....	....	....	....	....	....	....
Hardware manufacturing	1 969	1 051	897	480	1 023	545	49	26

* Population served is equal to or greater than that of industrial area.

that the plants manufacturing agricultural implements serve more persons than the population listed under "Remainder of State." A deficiency for this product however, exists within the State, and this justifies the inclusion of such plants in Table 7.

ECONOMIC BENEFITS RESULTING FROM CO-ORDINATED AGRICULTURAL AND INDUSTRIAL DEVELOPMENTS

The best way to visualize the financial improvement that would result from developing within the District the industries shown by Table 7 is through comparative statements showing the possible results of operating the District, first as an agricultural project alone, and then as a combined industrial and agricultural project. In preparing such statements, the gross annual value of the crops may be compared with the total value of the products given by Table 7. The gross values of the crops represent the prices at the land, before transportation and other charges have been added. The values of products likewise represent the prices at the factory. Both these values are measures of benefit to the agricultural and industrial workers as their individual net incomes should vary in proportion to gross values. It would be better, of

course, to base the comparison on the net income per capita before and after the introduction of industrial plants. Unfortunately, such an investigation would involve lengthy studies of agricultural and industrial production costs which vary so widely that generalizations would have little or no value.

In Table 9, the District is assumed to be fully developed. It must not be understood from this comparison that industrial workers would have nearly

TABLE 9.—COMPARATIVE ESTIMATED STATEMENTS OF ECONOMIC CHARACTERISTICS, 1929 BASIS

Description	Agricultural development	Industrial development	Total co-ordinated agricultural and industrial development
Population	25 000	100 000	125 000
Value of product	$13 000 000	$124 000 000	$137 000 000
Value of product per capita	$520	$1 240	$1 100

twice the average net income earned by agricultural workers. It would be reasonable to assume that about one-half the gross agricultural income per capita would be represented by production costs leaving a net average income per capita of $260. This would correspond to the average wage per capita of the industrial population which is estimated from Table 7 to be, $\frac{\$31\,691\,000}{100\,000}$, or $317.

With the improvement in agricultural prices that will follow recovery, there is every possibility that the incomes of agricultural workers and industrial workers would be about equal. It is certain that the industrial worker would have nothing to lose by devoting part of his time to gardening as the land utilized for raising truck yields a rate of return that would probably exceed industrial wage rates. On the other hand the worker has everything to gain by having an alternate source of employment during lay-offs and slack periods.

As indicated in Table 9 the population with combined industrial and agricultural developments would be at least five times that of the District if it were devoted to irrigation farming alone. This does not include the great number of additional people that would be necessary for professional and commercial occupations. Although this part of the population is difficult to estimate, it is certain that the increase would be substantial. Tables 1 to 9, therefore, do not include all the benefits of decentralization, industrial construction, highways, schools, churches, and buildings, because social institutions would all help to develop the population and would bring tremendous benefits to the District as a whole.

OPERATING PLAN FOR DECENTRALIZED INDUSTRIAL DEVELOPMENT

The assumption that 1 acre would be cultivated by each industrial worker is rather liberal for the needs of the average family and is greatly in excess of the actual requirements for a subsistence basis. The writers, however,

advocate an abundant rather than a subsistence level of living. The care of 1 acre of ground would require about 1.5 days per week per worker if the plots were operated according to the centralized plan as developed in Examples 1 and 2. As a matter of fact, industrial gardening on irrigation projects could be operated under no other plan, as it would be impractical to water and care for widely scattered projects.

Experience has shown that clearing and preparing the industrial gardens for planting could best be handled by the several companies within the District. The land adjacent to the plants could then be laid out in plots which would be assigned to the individual workers. The rotation plan of operation is not recommended as this removes the incentive derived from individual accomplishment.

Educational measures and strict supervision are absolutely essential. It would not be necessary for each plant to employ a trained agriculturist as one organization could handle the agricultural operations of several concerns. In addition to direction and education, the supervisor and his assistants would purchase and distribute seeds and also would manage the storage, distribution, and marketing of the agricultural products. Central storage facilities and distribution systems would be provided by each company. This measure is necessary to insure both the proper care and handling of the produce and a minimum of spoilage.

Conclusions

The foregoing detailed study indicates clearly that economic improvement would follow the co-operation of industry and agriculture on irrigated areas. It is not within the scope of a single paper, however, to present a quantitative analysis that even begins to show the potential benefits made possible by the great water conservation projects now (1936) being constructed.

New thinking is necessary if these results are to be achieved. One cannot view the areas affected by Boulder Dam, or by the works now (1936) being constructed in the Central Valley, the Columbia Basin, or the Tennessee Valley, in terms of flood control, power, or irrigation alone. Larger objectives will be attained if the new lands made available by each project are considered as a population base for a potential industrial development. It is entirely possible that many future industrial areas will develop about water conservation works rather than about congested cities.

These new industrial areas will not be limited by either State lines or other political boundaries; markets and transportation costs will determine the competitive limits within which the complementary industrial and agricultural developments of a water conservation project may serve economically. For example, the area which the future industries of the Central Valley project could serve may be much greater than that of the State of California as assumed in this investigation. To determine accurately the total potential industrial growth of the Central Valley, investigations should be made for the purpose of establishing the limits of the competing areas that will be adjacent to the Columbia Basin projects, to Boulder Dam, and to other

projects suitable for industrial development. Obviously, such investigations must have an important place in future national planning.

Widespread industrial decentralization correlated with water conservation can be brought about only by the joint efforts of business and the Federal Government. Manufacturers are turning to industrial decentralization principally because it is profitable. The opportunities for profitable industrial expansion will follow the extensive water-shed development programs that have been sponsored by the Government.

Industrial relocation in rural areas and subsequent co-ordination with agriculture will be beneficial to the entire country. The markets for both agricultural and industrial products will be brought closer together; therefore, decentralization will aid materially in the solution of the distribution problem. Existing industrial areas will not be affected adversely by such a movement; as a matter of fact, relief from over-congestion, due to the abandonment of many obsolete plants, accompanied by the migration of both the industry and its employees, would prove a positive benefit to many cities. Modern production units in healthful locations would replace many of the huge antiquated plants that are now found so frequently in industrial centers.

The stabilization of income through occupational diversification would result in a tremendous increase in demand for all products, and lower production costs would put these products within reach of the workers. It is safe to say, therefore, that the plants moving away from industrial cities would eventually create about them their own markets and would not injure or compete with the plants within existing industrial areas.

It is recognized generally that the present demand is not near the saturation point. What average person, for example, would not be glad to double his (or her) wardrobe if economic conditions permitted? A little thought will show that this situation extends into every department of modern life; a tremendous potential demand is thwarted by putting the products of industry just out of the wage earners' reach. Once these obstacles are overcome, it is safe to say that the total industrial production capacity of the nation will be found to be entirely inadequate.

Actual experience has proved that labor is ready and willing to accept both diversification and decentralization. This transition cannot be accomplished over night; considerable training and re-adjustment will be necessary before the benefits of decentralization may be realized. The end results, however, will more than justify the cost.

The diversification of labor would curtail the over-expansion of farm lands by eliminating from competitive use the areas devoted to industrial gardening. These gardens would not take away part of the existing market for agricultural produce; actually, they would help to lift the burden of relief which is not only robbing the farmer of his market, but increasing his tax load as well. The unemployed man cannot buy produce; it must be given to him, and the farmer now helps with the giving. It is safe to say that the greatest losses of the depression have been the production losses due to idle time. Any system that will help to capitalize idle time, will also reduce the relief load, and

thereby improve the economic condition of the entire country. Such improvement results inevitably in better markets for the products of both agriculture and industry.

To summarize, water conservation projects correlated with combined manufacturing and farming developments would assist in bringing about the balance between industrial and agricultural production that is necessary to economic stabilization.

Acknowledgment

This paper offers no brief for any formalized system of government, or controversial class philosophy. The suggestions herein contained are as applicable to the present industrial system and democratic form of government as to any other form. The paper has been kept as free from the mention of specific commercial names as possible in order to emphasize the purpose of the various examples as representative of their kind rather than as an effort to promote a certain isolated case.

Because of the importance of their contribution, however, free acknowledgment should be given to: Edward Hyatt, and Harry Barnes, Members, Am. Soc. C. E.; T. G. Graham, Vice-President of the B. F. Goodrich Company, of Akron, Ohio; A. E. Peters, General Superintendent of the Ingersoll-Rand Company, of Phillipsburg, N. J.; F. O. Hagie, Managing Secretary, Chamber of Commerce, Yakima, Wash.; Mr. F. Stewart Fitzpatrick, Manager, Construction and Civic Development Department, Chamber of Commerce, Washington, D. C.; Mr. Charles F. De Bardeleben, President, Alabama Fuel and Iron Company; Mr. John W. Haw, Director, Agricultural Development Department, Northern Pacific Railway Company, St. Paul, Minn.; Miss Mae A. Schnurr, Assistant to the Commissioner, U. S. Bureau of Reclamation, Washington, D. C.; Mr. John P. Ferris, Tennessee Valley Authority, Norris, Tenn.; C. D. Adams, General Superintendent, Holly Sugar Corporation, Sidney, Mont.; and J. J. Haly, Jr., Administrative Assistant, Department of Public Works, State of California, Sacramento, Calif. Fig. 1 of the paper was prepared from data supplied by research at the University of California. Unpublished data from the private files of the foregoing organizations, in large measure, form the basis for the paper.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

TAPERED STRUCTURAL MEMBERS: AN ANALYTICAL TREATMENT

Discussion

BY WALTER H. WEISKOPF AND JOHN W. PICKWORTH, ASSOC. MEMBERS, AM. SOC. C. E.

WALTER H. WEISKOPF[67] AND JOHN W. PICKWORTH,[67] ASSOC. MEMBERS, AM. SOC. C. E. (by letter).[67a]—The broad approach to the subject adopted by discussers of this paper is gratifying. It is a strong human trait to be distrustful of new methods and engineers are undoubtedly unfamiliar with the general application of substitute *I*-curves. The many suggestions and examples of use, as well as the fund of data that has been so generously contributed, are greatly appreciated, constituting as they do a tribute to the progressive spirit of the profession.

The frank statement that this method employs a substitute *I*-curve might lead some students of the subject to feel that for this reason the analysis is inherently inaccurate and to be distrusted. It should be realized that for all tapered members, except a few very simple forms, there is no exact treatment. Whether the designer realizes it or not, all methods are approximations or, in a sense, use substitute *I*-curves.

In the commonly used methods the integrations are performed by dividing the member into sections along its length. The value of $\frac{1}{I}$, or $\frac{M}{I}$, at the mid-point of each of these sections is assumed to apply to its entire length. In reality, this substitutes a series of steps, not only for the *I*-curves, but for the moment diagram as well. The latter inaccuracy is one which the writers' method does not introduce. The accuracy of the common method depends on the number of sections or steps employed. Usually, these sections are from 3 to 6 ft long, and the results are sufficiently accurate for ordinary engi-

NOTE.—The paper by Walter H. Weiskopf and John W. Pickworth, Assoc. Members, Am. Soc. C. E., was published in October, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1936, by Messrs. Fred L. Plummer, and LeRoy W. Clark; March, 1936, by Messrs. E. G. Paulet, J. Charles Rathbun, and Halvard W. Birkeland; May, 1936, by Messrs. C. W. Dunham, Fang-Yin Tsai, A. A. Eremin, and Austin H. Reeves; and August, 1936, by Messrs. A. W. Fischer, and L. Legens.

[67] Cons. Engr. (Weiskopf & Pickworth), New York, N. Y.

[67a] Received by the Secretary October 6, 1936.

neering purposes, but such approximations are no more trustworthy than the use of a smooth curve which follows the general course of the I-diagram.

Messrs. Large and Morris[53] have improved on the "stepped" substitute I-curve by introducing a series of inclined straight lines or chords. The area under the curve is thus divided into a series of trapezoids instead of rectangles. This method still involves a substitute I-curve, although an excellent one.

The accuracy of the method of substitute I-curves depends on the skill employed in fitting the curve. As Professor Clark discovered in turning his students loose on it, the method is not a fool-proof one that can be used safely by any beginner, but should be applied intelligently.

Mr. Reeves mentions two conditions that should be satisfied by the substitute curve: The area and the center of gravity of the area under the I-curve should be maintained. An additional thought should be borne in mind, namely, that where an exact fit is impossible it is more important to have a close fit at parts of the member where the moment of inertia is small rather than where it is large. This follows from the fact that the contribution of the work terms is large where the value of I is small, and *vice versa.*

Professor Plummer gives the relations between the terms F_1, F_2, and F_3 and the constants used in the conjugate and fixed-point methods of analysis. The following relations can be added to those which he gives:

$$u = \frac{F_1}{F_1 + F_3} \qquad (224)$$

and,

$$v = \frac{F_1}{F_1 + F_2} \qquad (225)$$

Professor Clark finds results obtained by means of the substitute I-curves satisfactory where he used the formulas for two-section members on a member which has two sections (Fig. 13(a)). In Fig. 13(b), however, he tried the formulas for two-section members on what is very obviously one with three distinct parts, a long uniform center section with sharp haunches at the two ends. It would have been more in the spirit of the writers' work to treat the center section as uniform and fit a substitute I-curve to each of the haunches. This can be done either by integrating with numerical values for an individual member, or by integrating to derive formulas. For such a member the formulas are no more complicated than those of Case 1 and are much more accurate.

Even if it is stretching the mathematics considerably to apply two-section formulas to a three-section member, the results are quite accurate if judgment is exercised. As mentioned previously, it is more important to have the two I-curves fit where the member is small than where it is large. This explains why Professor Clark obtained best results using a shape exponent which made the curves coincide where $x = 5$, and is obvious to one skilled in the method. Trying the other values ($x = 1$, $x = 2$, $x = 3$, etc.) was

[53] *Bulletin No. 66*, Ohio State Univ., Columbus, Ohio, November, 1932, p. 14.

misleading. On the other hand Mr. Reeves treating the same member (Fig. 27) immediately selected a value of 4.9 for *el*.

Professor Clark tried to test the speed of the method of substitute I-curves by having some of his students work the same problem, by using it, and by using the column analogy. The column analogy is a general method of analyzing statically indeterminate structures. The substitute I-curve affords a means of integrating the functions of x divided by I, which can be used in any method of analysis, the column analogy as well as any other. The two are not comparable. What Professor Clark means, of course, is that he compared the speed in obtaining results using formulas derived by the least work theory and substitute I-curves for the integrations, with the column analogy and a step-by-step summation for the integrations. Such a comparison in any case is not very clear, but considering that the students were presumably familiar with the step-by-step summation and totally unfamiliar with the method of substitute I-curves, it is of no value.

Mr. Paulet has contributed several illuminating examples of the use of the method. His thorough grasp of the fundamentals enables him to make variations in the application which add to the flexibility. For instance, his arbitrary assumption of the length in order to maintain a simple shape exponent is an ingenious hint that should prove useful to the skillful designer. Another excellent suggestion by Mr. Paulet to those who desire a visual guide in the selection of the shape exponent is to prepare a family of curves similar to Figs. 2 and 3 for ready use.

Professor Rathbun suggests another substitute curve, based on the assumption that the reciprocal of I varies as a parabola, and he further suggests that by increasing the power of x and the number of terms, the I-curves can be made to coincide at three, four, or more points. This is a possibility, of course, and there may be designers who prefer this curve. A disadvantage, however, is that the constants which Professor Rathbun designates p_1, p_2, etc., are expressed by formulas and have lost all obvious relationship to the form of the member. The constants, A (taper modulus) and n (shape exponent), in the writers' curve (Equation (1)), have a clear and obvious significance.

Professor Rathbun is incorrect in stating that the writers make the approximation of substituting a continuous curve for the two sections of the "stepped" member, Case 4. This case is treated by making the two parts of the member uniform, but not the same ($A = 0$, $B = 0$, I_A does not equal I_B). Since the terms, A, B, n, and m disappear, the formulas for Case 4 are not approximate as Professor Rathbun states, but are exact.

Mr. Birkeland stresses the importance of an understanding of the physical significance of the mathematical terms employed by the engineer. He presents a geometrical interpretation of the F-terms by means of angular changes in the unit beam due to various unit moments and loads, which is undoubtedly clarifying to those accustomed to study the behavior of structures in terms of slope changes and deflections.

Correctly, he points out that all F-terms beyond F_5 can be obtained from F_4 and F_5, but such formulas as his Equation (165), should be used with

care. In members of more than one section the integrations must be performed separately for each section.

Mr. Birkeland suggests using I_C instead of I_A as a reduction factor to obtain the F-terms. This is entirely a matter of personal preference. The writers intended to present a general method which an individual engineer can use to best advantage by adaptation to his own particular taste. If a designer prefers using I_C to I_A, and has sufficient mastery of the method to make this modification throughout, there can be no objections.

Mr. Dunham very properly emphasizes the importance of fixing the working lines correctly. The suggestions embodied in Fig. 12, and in Professor Tsai's Fig. 26 should be followed. Another method, brought out by Mr. Fischer in connection with the curved girder of a rigid frame, is to use a working line through the center of gravity of the ends of the member and later apply a correction.

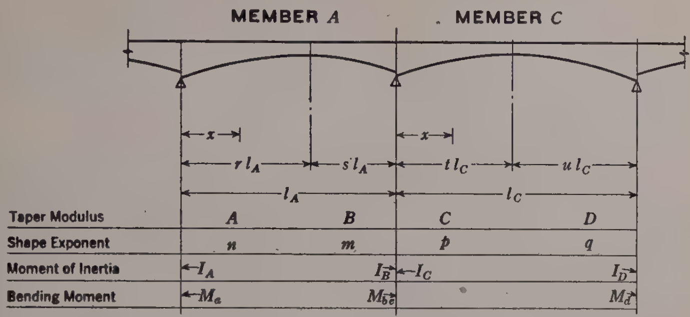
Professor Tsai presents an extensive discussion of the various methods of analyzing structures composed of tapered members, and very complete tables showing the equivalents of the constants of the various systems. These are valuable records compiled in a scholarly manner. He is to be complimented on his presentation of the generalized equation of three moments. The method so often used in the past for want of a better one, assuming that a tapered continuous beam is uniform for the purpose of computing the reactions, is frequently grossly inaccurate. There has been a real need for this equation, but to the best of the writers' knowledge it has never appeared in any text. The writers had, independently, derived a generalized three-moment equation for tapered members using F-terms (see Fig. 32).

Professor Tsai tried the method of substitute I-curves on the member shown in Fig. 25[53] for a concentrated load at Point $k = 0.7$ and found that the resulting fixed-end moments differed about 12% from those obtained by step-by-step summation. A better test for accuracy is the area under the influence line, or what amounts to the same thing, the fixed-end moment for uniform load. This will more nearly approach the actual loading on the member.

For a uniform load the difference between the step-by-step value and that obtained using Professor Tsai's substitute I-curve is 6 per cent. Professor Tsai's shape exponent ($m = 0.64$), however, is poor. He seems to have sensed this for he states that better values might have been obtained using $f = 0.1$. Using a value of $m = 1$ the difference for the concentrated load is 5% instead of 12%, and for the uniform load, 2% instead of 6 per cent. If he had continued his comparative studies in order to develop influence lines by both methods, he would have appreciated the savings in time and labor which accrue once F_1, F_2, and F_3 are found.

Apparently, Professor Tsai does not fully appreciate the inaccuracy of the step-by-step summation. For M_A on this same member he obtains 27.26 ft-kips, whereas Professor Large obtained 27.95, a difference of $2\frac{1}{2}\%$; yet Professor Tsai uses his step-by-step summation as a standard of comparison for the substitute I-curve. The writers obtained a value of $M_A = 28$ ft-kips using $m = 1$. Professor Tsai refers to the writers' "exact" formulas. These

formulas are not exact, and in no place are they so designated. An exception to this generality is Case 4, the "stepped" member (and, of course, Case 5), the formulas for which are exact.



$$\frac{M_a l_A F_{1A}}{I_A} + M_{bc}\left(\frac{l_A F_{2A}}{I_A} + \frac{l_C F_{3C}}{I_C}\right) + \frac{M_d l_C F_{1C}}{I_C} = \begin{cases} -\dfrac{P l_A^2 F_{4A}}{I_A} \\[2ex] -\dfrac{P l_A^2 F_{6A}}{I_A} \\[2ex] -\dfrac{P l_C^2 F_{5C}}{I_C} \\[2ex] -\dfrac{P l_C^2 F_{7C}}{I_C} \\[2ex] -\dfrac{w_A l_A^3 F_{8A}}{I_A} - \dfrac{w_C l_C^3 F_{9C}}{I_C} \end{cases}$$

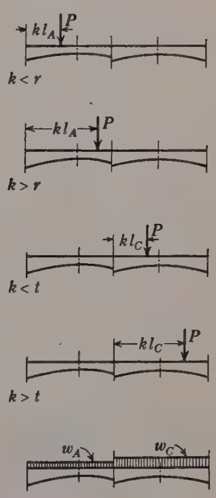


FIG. 32.—THREE-MOMENT EQUATION FOR TAPERED MEMBERS

Professor Tsai suggests another substitute I-curve in the form, $I_x = a + bx^2 + cx^3$; but in order to integrate functions of x divided by I, the value of I must be expressed in some reciprocal relation to x. The writers do not understand how Professor Tsai's curve makes it possible to perform these integrations; nor does he furnish any light in this direction.

Professor Tsai mentions the condition at the intersections of members. When members are shallow this presents no difficulties, but when they are deep and the working points are well within the adjacent members, the point is well worth considering. The most rational treatment of this condition is to assign a value of infinity to the moment of inertia for the length between the edge of an adjacent member and the working point. Thus, the columns in Fig. 26(a) would have an infinite moment of inertia for h'', the upper part of the length, h. This can easily be taken care of in the writers' formulas by

considering the column as a two-section "stepped" member (Case 4) with I_B equal to infinity. The resulting formulas, of course, are quite simple.

Mr. Eremin gives a simple method of computing the F-terms which is convenient for members of one section. It is not applicable, however, to multi-section members. Mr. Fischer presents an interesting application of the substitute I-curves to the rigid frame bridge. Although the writers differ with him in the selection of working lines they have found by various checks, as he has, that the results are accurate.

Mr. Legens presents Müller-Breslau's analysis of an arch, which utilizes a substitute I-curve. This is a good example of a use of the method beyond any mentioned by the writers. It should be emphasized that their purpose was to furnish, not a list of formulas but a method, general in its scope. which has many applications beyond the few mentioned.

For those who have repetitive cases to handle, tables are most useful and as has been pointed out by more than one discusser, the method will be an aid to those computing and checking such tables. The writers do not agree that design should be restricted in any way by the scope of existing tables. It is desirable to have available a method (or rather, methods) for analyzing any and all shapes.

Mr. Birkeland and Professor Tsai refer to the importance of understanding the physical meaning of the F-terms. They interpret these constants in terms of angle changes. Three discussers, Professor Plummer, Mr. Birkeland, and Professor Tsai present conversion tables which show the equivalents of the F-terms to the constants of other methods. The writers agree that the physical meaning of such constants is important to a thorough understanding of the subject. The following analytical interpretation of the meaning of the F-terms is preferred by the writers.

The F-terms are ratios, the integrals being divided by $\frac{l^3}{I_A}$ to obtain them. This simplifies numerical computation. For theoretical purposes, however, it is better to denote each integral as it stands by a symbol. Using G then instead of F:

$$G_1 = \frac{l^3 F_1}{I_A} = \int \frac{x\,(l - x)\,dx}{I_x} \quad \ldots\ldots\ldots\ldots (226)$$

$$G_2 = \frac{l^3 F_2}{I_A} = \int \frac{x^2\,dx}{I_x} \quad \ldots\ldots\ldots\ldots (227)$$

and,

$$G_3 = \frac{l^3 F_3}{I_A} = \int \frac{(l - x)^2\,dx}{I_x} \quad \ldots\ldots\ldots\ldots (228)$$

It is apparent from Equations (226), (227), and (228), that G_2 is the moment of inertia of the reciprocal I-diagram about an axis at the left end of the member, and G_3 is the moment of inertia of the reciprocal I-diagram about an axis at the right end; G_1 is a fourth dimensional quantity for which there seems to be no name in engineering nomenclature. It is the summation of the product of each elementary strip of the area under the reciprocal I-curve

times, first, its distance from the left end, and then its distance from the right end. It is like a product of inertia, except that the two axes are parallel instead of at right angles to each other. It will, therefore, be termed the "parallel product of inertia."

Other properties of the reciprocal I-diagram can be expressed in terms of the constants, G_1, G_2, and G_3, thus:

$$A_o = \int \frac{dx}{I_x} = \frac{2\,G_1 + G_2 + G_3}{l^2} \quad \text{.......} \quad (229)$$

$$\int \frac{x\;dx}{I_x} = \frac{G_1 + G_2}{l} \quad \text{.......} \quad (230)$$

$$\int \frac{(l - x)\;dx}{I_x} = \frac{G_1 + G_3}{l} \quad \text{.......} \quad (231)$$

$$x_o = l\,\frac{G_1 + G_2}{2\,G_1 + G_2 + G_3} \quad \text{.......} \quad (232)$$

and,

$$v_o = l\,\frac{G_1 + G_3}{2\,G_1 + G_2 + G_3} = l - x_o \quad \text{.......} \quad (233)$$

Equation (229) is the area under the reciprocal I-curve; Equations (230) and (231) are the statical moments of the area under the reciprocal I-curve about the left end and the right end, respectively; and x_o and v_o, respectively (Equations (232) and (233)), are the distances to the center of gravity from the left and right ends of the member.

The moment of inertia of the reciprocal I-curve about its center of gravity axis is found to be:

$$G_o = \frac{G_2\,G_3 - G_1^2}{2\,G_1 + G_2 + G_3} \quad \text{.......} \quad (234)$$

It is interesting to note the appearance of the expression, $G_2 G_3 - G_1^2$, analogous to $F_2 F_3 - F_1^2$ which appeared in the denominator of the equations for fixed-end moments and for stiffness.

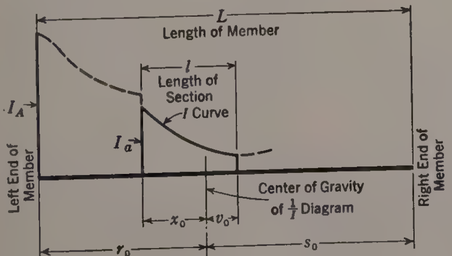


FIG. 33

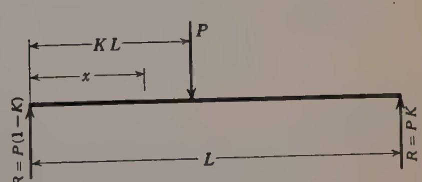


FIG. 34

Since G_2 and G_3 are moments of inertia, their values can be transferred to different axes as is done with ordinary moments of inertia. This suggests

that in a complicated member composed of several sections the values of G_2 and G_3 can be summed up as the area of each section, the location of its center of gravity, and its G_0 are known.

Fig. 33 represents the I-curve for a section of a member. In general, small letters will be used to represent properties of the section about axes of the section, and capital letters to represent properties of the member about its axes. The values of g_1, g_2, and g_3 for this section about its own ends can be found by the formulas of Case 3, the section of single taper, multiplying the F-terms by $\frac{l^3}{I_a}$. Then, from Equations (229) to (234),

$$a_0 = \frac{2\,g_1 + g_2 + g_3}{l^2} \qquad \text{(235)}$$

$$x_0 = l\,\frac{g_1 + g_2}{2\,g_1 + g_2 + g_3} \qquad \text{(236)}$$

and,

$$g_0 = \frac{g_2\,g_3 - g_1^2}{2\,g_1 + g_2 + g_3} \qquad \text{(237)}$$

Knowing x_0, values of r_0 and s_0 can be found. Then for the entire member,

$$G_2 = \Sigma\,(g_0 + a_0\,r_0^2) \qquad \text{(238)}$$

and,

$$G_3 = \Sigma\,(g_0 + a_0\,s_0^2) \qquad \text{(239)}$$

An expression analogous to these for G_1 is:

$$G_1 = \Sigma\,(-\,g_0 + a_0\,r_0\,s_0) \qquad \text{(240)}$$

Thus the properties of a member can be found simply no matter into how many sections it is divided. For a concentrated load, P (see Fig. 34), the terms, G_4 and G_5, analogous to F_4 and F_5, can similarly be found. To the left of the load the simple beam moment is:

$$M_s = P\,(1 - K)\,x \qquad \text{(241)}$$

and to the right of the load the simple beam moment is:

$$M_s = PK\,(L - x) \qquad \text{(242)}$$

Then

$$G_4 = \frac{P\,L^3\,F_4}{I_A} = \int_0^L \frac{M_s\,x\,dx}{I_x} = \int_0^{KL} P\,(1 - K)\,\frac{x^2\,dx}{I_x} + \int_{KL}^{L} P\,K\,x\,(L - x)\,\frac{dx}{I_x} \qquad \text{(243)}$$

By Equations (238), (239), and (240):

$$G_4 = P\,(1 - K) \sum_{0}^{KL} (g_o + a_o\, r^2_o) + P\,K \sum_{KL}^{L} (-\,g_o + a_o\, r_o\, s_o) \ldots (244)$$

In a similar manner,

$$G_5 = \frac{P\,L^3\,F_5}{I_A} = P\,(1 - K) \sum_{0}^{KL} (-\,g_o + a_o\, r_o\, s_o) + P\,K \sum_{KL}^{L} (g_o + a_o\, s^2_o) \ldots\ldots\ldots\ldots\ldots\ldots (245)$$

Thus, all the elastic properties can be obtained for a member of any number of sections (see Table 9).

TABLE 9.—PROPERTIES OF RECIPROCAL *I*-DIAGRAM FOR A SECTION OF A MEMBER

INTEGRAL	DEFINITION	FORMULA
$\int \frac{dx}{I_x}$	AREA under $\frac{1}{I}$-curve	$\frac{2\,g_1 + g_2 + g_3}{l^2}$
$\int \frac{x\,dx}{I_x}$	STATICAL MOMENT about left end of section	$\frac{g_1 + g_2}{l}$
$\int \frac{(l - x)\,dx}{I_x}$	STATICAL MOMENT about right end of section	$\frac{g_1 + g_3}{l}$
$\int \frac{x\,(l - x)\,dx}{I_x}$	PARALLEL PRODUCT OF INERTIA	$g_1 = \frac{l^3\,F_1}{I_a}$
$\int \frac{x^2\,dx}{I_x}$	MOMENT OF INERTIA about left end of section	$g_2 = \frac{l^3\,F_2}{I_a}$
$\int \frac{(l - x)^2\,dx}{I_x}$	MOMENT OF INERTIA about right end of section	$g_3 = \frac{l^3\,F_3}{I_a}$
	Distance from left end of section to center of gravity	$l\,\frac{g_1 + g_2}{2\,g_1 + g_2 + g_3} = x_o$
	Distance from right end of section to center of gravity	$l\,\frac{g_1 + g_3}{2\,g_1 + g_2 + g_3} = v_o$
	MOMENT OF INERTIA about axis through center of gravity	$\frac{g_2\,g_3 - g_1^2}{2\,g_1 + g_2 + g_3} = g_o$

This theory of combining sections of a member has been found particularly convenient where a part of the member is uniform, since for this section the *g*-terms are very simple. From the formulas given in Case 5, the uniform member, there is obtained:

$$g_1 = \frac{l^3}{6\,I_a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (246)$$

$$g_2 = g_3 = \frac{l^3}{3\,I_a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (247)$$

and,

$$g_o = \frac{l^3}{12\,I_a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (248)$$

Equations (247) and (248) are obviously the moments of inertia of a rectangle of length, l, and width, $\frac{1}{I_a}$, about axes at its edge and center of gravity.

In the member given by Professor Clark shown in Fig. 13(b) (a uniform center section with a straight haunch at each end) this method was found more convenient than the three section formulas. It was also found possible to integrate the section with the straight haunch using the actual I-curve, $I_x = (a + bx)^3$, to find g_1, g_2, and g_3. The foregoing relations thus make an exact solution possible for a member of any number of sections where the moment of inertia varies as the cube of the depth and the edges of each section are straight.

It has even been found possible to perform the integrations and thus obtain an exact solution where the contours of portions of a member are conic sections and I varies as the cube of the depth, $I_x = (a + bx + cx^2)^3$, but this solution is too long for use in design work.

It is apparent that all the relations of the properties of the reciprocal I-curve given in Equations (226) to (248) are mathematical theorems that do not depend on, and are not limited by, substitute I-curves. In the original paper it was brought out that the substitute I-curve furnishes a way of integrating the functions of x divided by I which can be used in any method of analysis. Conversely, it is possible to use these theorems without employing substitute I-curves. Thus, if one chose, one could find the values of g_1, g_2, and g_3 of the sections of a member by the step-by-step summation or by the Large-Morris approximation, and then, by means of the relations of Equations (235) to (245), find all the elastic properties of the member.

It has been found that many different kinds of members can be treated more conveniently by summing the properties of individual sections than by endeavoring to consider the member as a whole in the beginning. Examples of such members are:

Example (a).—A member framing into two deep members of such proportions that an infinite moment of inertia should be used at both ends. There would be three or more sections to such a member. The numerical work, however, is greatly reduced by the fact that for the end sections, in which I equals infinity, the g-terms are zero.

Example (b).—Steel members in which cover-plates discontinue at frequent intervals. In some cases it is accurate enough to use a smooth substitute I-curve for the entire length. In others, any number of sections can be employed so that the results can be obtained with any desired degree of accuracy. This is an example of the flexibility of the general method.

The writers recognize fully the thought and careful work which has gone into the preparation of the many discussions of this paper. They wish to thank all the contributors most sincerely, and express appreciation of the manner in which they have amplified a subject that was treated, with some difficulty, within the confines of limited space.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SUCCESSIVE ELIMINATION OF UNKNOWNS IN THE SLOPE DEFLECTION METHOD

Discussion

JOHN B. WILBUR, ASSOC. M. AM. SOC. C. E.

JOHN B. WILBUR,[40] ASSOC. M. AM. SOC. C. E. (by letter).[40a]—The valuable additions that have been contributed by the discussers of this paper are appreciated. In one general respect the method of presenting the solution of a continuous beam by successive elimination of unknowns was unfortunate in that it was included as a pedagogical step in leading the reader gradually into the more complicated problem of a building frame, in which it was believed the proposed expedient might be of value. The writer agrees with those who suggest that other methods of solution are likely to be superior in analyzing continuous beams.

A method of determining, approximately, the bending moments in the members meeting at a particular joint of a building frame, is presented by Mr. Willson. He assumes that there is no side-sway in the building. Under loadings where side-sway would occur, his Equation (19) would be incorrect unless revised to include this effect. Such revision would seem to lead to a more cumbersome expression.

Mr. Andersen believes that the method of eliminating unknowns suggested by the writer would immediately suggest itself to one setting up the solution of a problem by the slope deflection method. Judging from his own experience, the writer is forced to disagree on this point. Mr. Andersen states moreover, that, in his opinion, designers do not object to the solution of simultaneous equations. Although this may be true, those who pay for the designer's time must certainly have an opposite point of view, especially if a large number of unknowns are involved.

NOTE.—The paper by John B. Wilbur, Assoc. M. Am. Soc. C. E., was published in December, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1936, by Messrs. C. A. Willson, Paul Andersen, and R. W. Stewart; April, 1936, by Adolphus Mitchell, Jun. Am. Soc. C. E.; May, 1936, by Messrs. Fang-Yin Tsai, A. Floris, A. W. Fischer, and L. E. Grinter; and August, 1936, by L. T. Wyly, M. Am. Soc. C. E.

[40] Asst. Prof. of Civ. Eng., Mass. Inst. Tech., Cambridge, Mass.

[40a] Received by the Secretary September 29, 1936.

The necessity of understanding the sign convention fully in using the slope deflection method, is emphasized by Mr. Stewart, who gives an excellent statement of these conventions. Professor Grinter likewise shows, with examples, the conventions which the writer has followed. Both these discussions should be of value to those who may have difficulty with this important aspect. The modification of the writer's method as applied to a building frame, which Mr. Stewart suggests, is of genuine interest. Professor Grinter stresses the need of a physical conception of structural problems; and with this belief, the writer is in full accord.

Mr. Mitchell states that office practice should be such that, in so far as is practicable, it is self-checking. There can be no dispute on this point. Longer methods which have this advantage are often preferable to shorter approaches which leave one in doubt as to the accuracy of one's result. The method suggested by the writer, however, is self-checking. It yields certain values of R and θ, so that the solution corresponds to a possible deformation of the structure. The solution is thus consistent elastically. If the moments from these functions are in static equilibrium with the external forces acting on the structure, the solution is also consistent statically, and must be correct.

The equations contributed by Professor Tsai which take into account beams of variable moment of inertia, are of value in the general slope deflection solution and, of course, aid in the particular technique of solution suggested by the writer.

Mr. Fischer raises an interesting problem with respect to the solution by slope deflection of a frame with inclined legs. Inasmuch as the treatment of this problem as given in certain engineering literature is incorrect, in the writer's opinion, the treatment by Mr. Fischer is valuable.

The outline of the development of the slope deflection method given by Mr. Wyly is excellent, and will be of value to those investigating this field. When his discussion is printed in *Transactions*, with the paper and other discussions, Mr. Wyly will add the following after Step (9): The method of successive elimination of unknowns in the solution of secondary stress problems was used at least as early as 1911 by David A. Molitor, M. Am. Soc. C. E.[35a] Mr. Molitor solved twenty-two equations for twenty-two moments, all expressed in terms of the first two unknown moments.

[35a] "Kinetic Theory of Engineering Structures," by David A. Molitor, 1911, pp. 232 *et seq.*

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

REINFORCED CONCRETE MEMBERS UNDER DIRECT TENSION AND BENDING

Discussion

BY MESSRS. ODD ALBERT, AND D. B. GUMENSKY

ODD ALBERT,[30] ASSOC. M. AM. SOC. C. E. (by letter).[30a]—The statements made in this paper are correct, but the author has not treated the complete problem. There are three general cases in which part of the section is in compression; namely: (*a*) Tension steel only; (*b*) tension steel and compression steel; and (*c*) tension steel and compression steel alike.

The author treats only Case (*a*) in which steel is placed in the tension side. This produces a design in which steel acts in tension and concrete in compression, and results in some rather complicated diagrams, which involve eleven operations for each solution. He also states that, should the intersection of the curves fall outside the limits of the chart, a re-design may be necessary.

The problem can be simplified considerably, and a chart, such as Fig. 12, can be produced with only straight lines, that will take care of all conditions.

Effective Depth.—Referring to Fig. 2 and taking moments about the center of the steel (see Fig. 13):

$$T\left(e - d + \frac{d_t}{2}\right) = \frac{f_c\, b\, d\, k}{2}\left(d - \frac{k\, d}{3}\right) \quad \text{(58)}$$

the moment, M_s, with reference to the tension steel is:

$$M_s = T\left(e - d + \frac{d_t}{2}\right) \quad \text{(59)}$$

or, $M_s = 0.5\, b\, d^2\, k\, j\, f_c$. Solving for d:

$$d = \sqrt{\frac{M_s}{K\, b}} \quad \text{(60)}$$

NOTE.—The paper by D. B. Gumensky, Assoc. M. Am. Soc. C. E., was published in December, 1935, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: March, 1936, by Messrs. A. W. Fischer, William E. Wilbur, and F. C. Snow; April, 1936, by Messrs. Ralph E. Byrne, Jr., Carl H. Heilbron, Jr., F. E. Turneaure, H. E. Warrington, William A. Larsen, and B. Kovediaeff; and August, 1936, by Messrs. A. Floris, and David B. Hall.

[30] Faculty Lecturer, Advanced Design of Structures, New York Univ.; Engr., Brick Mfrs. Assoc. of New York, New York, N. Y.

[30a] Received by the Secretary September 25, 1936.

Equation (60) is identical with the beam formula for tension steel only. The difference is that the moment, M_s, is taken with reference to the tension

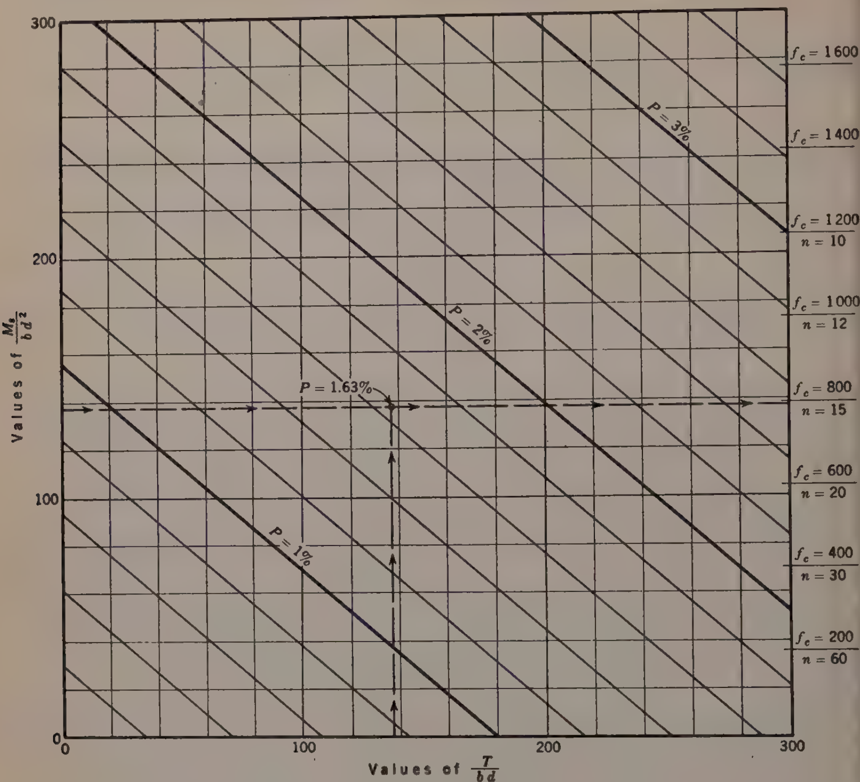


FIG. 12.—COLUMN IN BENDING AND TENSION

steel instead of to the center of the section. Therefore, after the change of M to M_s, ordinary beam tables can be used to find the dimensions, b and d.

Tension Steel.—Writing a projection equation:

$$A_s\,f_s - T - \frac{b\,k\,d}{2}\,f_c = 0 \quad \dots\dots\dots\dots\dots\dots (61)$$

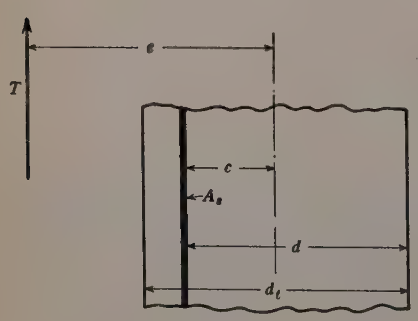


FIG. 13

and, solving for A_s,

$$A_s = \frac{b\,k\,d\,f_c}{2\,f_s} + \frac{T}{f_s} \quad \dots\dots (62)$$

Since $\frac{b\,k\,d\,f_c}{2\,f_s} = \frac{M_s}{f_s\,j\,d}$:

$$A_s = \frac{M_s}{f_s\,j\,d} + \frac{T}{f_s} \quad \dots\dots (63)$$

The second term of Equation (63) is identical with the formula for the steel area for a beam with tension steel

only. It must be remembered that the moment, M_s, refers to the center of the tension steel.

Example A-1.—Assume the same values as in Example 1, namely, that $T = 16\,530$ lb; $e = 14$ in.; $d = 10$ in.; $d_c = 2$ in.; $f_c = 800$ lb per sq in.; $f_s = 18\,000$ lb per sq in.; $K = 138.7$; and $j = 0.867$: Hence, $M_s = 16\,530 \times 14\text{–}4 = 165\,300$ in-lb. Therefore, the minimum depth for the allowable stresses will be, by Equation (60): $d = \sqrt{\dfrac{165\,300}{138.7 \times 12}} = 9.93$ in.

The selection of 10 in. will correspond to a somewhat lower concrete stress. The author gives $f_c = 795$. The tension steel will be: $A_s = \dfrac{165\,300}{18\,000 \times 0.867 \times 10} + \dfrac{16\,530}{18\,000} = 1.97$, which is correct, although a little higher than the value obtained by the author.

Plotting the Chart.—Equation (63) indicates that the most economical section is obtained by using the maximum allowable steel stress. Therefore, it is scarcely necessary that variable steel stresses be introduced on a chart. It is better to begin with the allowable steel stress, and to let the concrete stress vary. From Equation (59):

$$\frac{M_s}{b\,d^2} = \frac{f_c\,k}{2}\left(1 - \frac{k}{3}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (64)$$

Introducing the quantity, $n\,f_c = 12\,000$ (that is, $n = 10$ for $f_c = 1\,200$; $n = 12$ for $f_c = 1\,000$, etc.), Equation (64) yields $k = 0.4$. Hence,

$$\frac{M_s}{b\,d^2} = 0.1733\,f_c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (65)$$

Also, for $k = 0.4$, $f_s = 18\,000$, and $A_s = p\,b\,d$, Equation (63) yields:

$$18\,000\,p = 0.20\,f_c + \frac{T}{b\,d} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (66)$$

and if f_c is eliminated from Equations (65) and (66):

$$15\,600\,p = \frac{M_s}{b\,d^2} + 0.8667\,\frac{N}{b\,d} \quad \ldots\ldots\ldots\ldots\ldots\ldots (67)$$

Equation (67) gives the relation between p, M_s, and N. It will be noted that this formula represents a straight line, and, therefore, two points only are necessary to determine this line. For example, for $p = 1\%$:

$$156 = \frac{M_s}{b\,d^2} + \frac{26}{30}\,\frac{N}{b\,d} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (68)$$

For $M_s = 0$, $\dfrac{N}{b\,d} = 180$ in Equation (68); and, for $N = 0$, $\dfrac{M_s}{b\,d^2} = 156$. These two points connected, establish the line representing $p = 1\%$, etc.

Equation (65) demonstrates that the concrete stress varies in direct relation to the $\frac{M_s}{b\,d^2}$ - values, and, therefore, that they can be plotted on the right side of Fig. 12. These values represent Equation (60) graphically.

Example A-2.—Referring again to the data in Example 1: As before, $M_s = 165\,300$ in-lb, and $\frac{M_s}{b\,d^2} = \frac{165\,300}{12 \times 100} = 137.7$. Similarly, $\frac{N}{b\,d} = \frac{16\,500}{12 \times 10} = 137.7$. The intersections of these two lines (see the broken dotted lines in Fig. 12) give $p = 1.63\%$ and $f_c = 795$, which checks with previous results.

D. B. Gumensky,[31] Assoc. M. Am. Soc. C. E. (by letter).[31a]—Many additional data and a further development of the subject were brought to light by those who discussed this paper. With the advancements made recently in the analyses of continuous frames and with the generally increasing scale of construction, particularly of hydraulic structures, the problem of designing reinforced concrete members subjected to combined stress of bending and direct tension is becoming more and more important.

By taking the moments about the tension steel, the problem can be divided into two distinct steps: (1) Determining the section to resist this moment; and (2) determining the additional steel in the tension face required by the direct tension; in case of direct compression, the additional amount is negative. This fact was demonstrated very ably in different ways by Messrs. Fischer and Wilbur.

Professor Snow takes the matter of design for compressive and tensile reinforcement and presents a complete solution of the problem. In connection with this question the writer would like to call attention to the fact that the seemingly simple solution of his original Case 1 in Fig. 1 may become more complicated when the direct pull is within the section, but still close enough to the tension steel to produce compression in concrete outside the farther steel layer. This situation recently presented itself to the writer in the interpretation of the experimental measurement of a double reinforced member subjected to a combined stress of direct tension and bending. Fig. 14 illustrates this condition which overlaps Case 1.

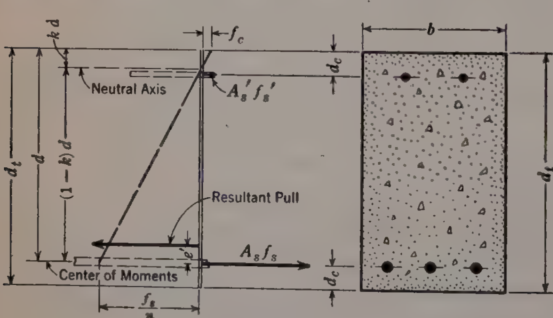


Fig. 14

Mr. Byrne treats the general problem of a direct stress and bending with very gratifying results. Mr. Heilbron, by clever modifications of the original charts and formulas, produces a new chart (Fig. 8) that has the advantage of

[31] Engr., Metropolitan Water Dist. of Southern California, Los Angeles, Calif.
[31a] Received by the Secretary October 2, 1936.

definite intersections and increased general usefulness, as it can be used for any value of n. In his brief discussion Dean Turneaure contributes toward a clearer understanding of the problem. The "cut-and-try" method of solution mentioned by him seems to be favored by many designers.

The solution and the references presented by Mr. Warrington are of interest and were entirely unknown to the writer. Mr. Larsen should be congratulated on his elaborate and excellent discussion. His Fig. 9, and the charts presented by Mr. Wineland as shown in Fig. 10, should be of great interest to those interested in this broad problem.

The writer wishes to acknowledge his gratitude to Mr. Kovediaeff for his discussion and for his assistance in the preparation of the paper. He also appreciates the discussion by Mr. Floris and sincerely hopes that the comments thus offered will provide a stimulus for further investigations and a better understanding of reinforced concrete problems.

Thinking in terms of the physical action of structures and structural members rather than in mathematical terms, is what the designer strives to do. Mr. Hall recapitulates the problem of combined stresses in its true physical meaning. His discussion illuminates the aspects of the problem not covered by other writers, particularly the solution for a minimum total area of steel in a member reinforced in both faces.

The chart constructed by Mr. Albert possesses the advantage of extreme simplicity in construction and in use. This method and chart give a quick, simple means of designing a new member for a given condition of stresses. In analyzing an existing structural member, one would need to resort to "trial and error", or should have a series of such charts constructed for several unit stresses in steel covering a range, say, from 1 000 lb per sq in. to 25 000 lb per sq in.

Acknowledgments.—In closing, the writer wishes to express his appreciation of the interest displayed by those who commented on his paper. He also wishes to state that the material contained in the discussion represents a valuable contribution to engineering literature, and, in this light, his original presentation now seems justified. The paper had been prepared in connection with the design of hydraulic structures in the Colorado River Aqueduct and was submitted to the University of Southern California in partial fulfillment of requirements for the degree of Master of Science.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

TALL BUILDING FRAMES STUDIED BY MEANS OF MECHANICAL MODELS

Discussion

BY FRANCIS P. WITMER, AND HARRY H. BONNER, ESQ.

FRANCIS P. WITMER,[24] M. AM. SOC. C. E., AND HARRY H. BONNER,[25] ESQ. (by letter).[25a]—The writers are gratified at the satisfactory verification of their results on the part of several who have discussed this paper. It must be borne in mind that their intent was not to produce exact quantitative values, but only to determine trends of reaction ratios resulting from assumed variations in the design of the bents. Agreement with the writers' results is particularly noticeable in the two upper curves of Fig. 11, in the discussion by Professor Maugh, and also in the celluloid model tests of Mr. Morrison. Professor Tsai obtains very interesting relations for single-story bents, which are also confirmatory of the writers' results, and Mr. Fischer shows a similar agreement with theory.

The failure of total upward and downward reactions to be equal in some cases, as mentioned by Mr. Morrison, is not surprising, since actual readings were recorded in all cases without any attempt, arbitrarily, to correct them in order to produce the agreement demanded by statics. It is believed that, in most cases, the discrepancies will not be great. The model has since been improved by the use of positive metal clamps, tightened by screws, instead of the wooden joint blocks, and with these the equality of upward and downward reactions is much more nearly obtained in all tests. It is interesting to note, however, that, with these metal connections, repetition of tests previously made with wooden blocks produces reaction ratios in very close agreement with those first found, thus indicating a surprising degree of efficiency on the part of the wooden blocks.

The discussion by Professor Large and Mr. Carpenter is very interesting and to a considerable extent confirmatory. It is felt, however, that they have

NOTE.—The paper by Francis P. Witmer, M. Am. Soc. C. E., and Harry H. Bonner, Esq., was published in January, 1936, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: April, 1936, by Messrs. L. C. Maugh, L. J. Mensch, and Gilbert Morrison; August, 1936, by Fang-Yin Tsai, Assoc. M. Am. Soc. C. E.; and September, 1936, by Messrs. George E. Large and S. T. Carpenter, Jr., and A. W. Fischer.

[24] Director, Civ. Eng., Univ. of Pennsylvania, Philadelphia, Pa.

[25] Instr. in Mech. Drawing, Frankford High School, Philadelphia, Pa.

[25a] Received by the Secretary October 14, 1936

expected rather too great a degree of exactness in some cases where they find what they term "serious discrepancies." The extension of their study to cases with relatively heavy columns is not necessarily contradictory of the writers' results. In subsequent tests, the writers have found that, when all columns are increased in size, but are still kept equal to each other, the reaction ratio will be increased algebraically. Some results in this connection are indicated in Table 4. The last ratio, + 23%, for Case *C*-2 with heavier columns, compares well with + 20% for Case 11-*C*-2 in Table 3.

TABLE 4.—COMPARISON OF REACTION RATIOS

Case	Reaction Ratios, $\frac{R_i}{R_o}$	
Relative column areas	1	2
Relative moments of inertia	1	8
$A-1$	−15%	0%
$C-1$	+76%	+92%
$C-2$	0%	+23%

Professor Large and Mr. Carpenter state that, in computing the stiffness of the members, reduced lengths of girders ("Spurr lengths") were used instead of center-to-center lengths, as they ordinarily "gave results nearer to experimental values." In their original report[17] they state that the use of "center-to-center lengths of members in computing stiffness corresponds more nearly to the actual performance so far as girder shears are concerned." These two statements appear to be contradictory. The improved metal connections used by the writers in their later tests approximate closely the center-to-center length relation, whereas the original wooden blocks caused a reduced length of all members. The close agreement in results obtained with the two types of connections made it appear that the reaction ratios were not greatly affected by the shortened lengths. If further model study could demonstrate that this is true, the use of center-to-center lengths would greatly simplify the determination of reactions. Of course, reduced lengths are necessary in computing moments in girders.

The writers would not expect their conclusions to apply to relations not reasonably within the range comprehended in their tests. Considerable increase in the size of columns might naturally call for some modification. They believe, however, that the qualitative trends which they endeavored to determine have been well substantiated by the various discussions.

The reference by Professor Large and Mr. Carpenter to the impropriety of applying "20-story methods to 50-story towers", of course, is unquestioned, as is also the proper consideration of deflection in the case of tall towers. The writers' models, in some cases, had a height-to-width ratio as great as 8.7 to 1, and were thus properly in the tower class. The number of stories is not so significant as the height-to-width ratio. No attempt was made to

[17] "Tests and Design of Steel Wind Bents for Tall Buildings," by George E. Large, Assoc. M. Am. Soc. C. E., Samuel T. Carpenter, Jun. Am. Soc. C. E., and Clyde T. Morris, M. Am. Soc C. E., *Bulletin No. 93*, Appendix, Eng. Experiment Station, Ohio State Univ., Columbus, Ohio.

determine whether considerations of deflection should demand one method of computation rather than another in order to maintain a maximum degree of economy. The purpose was to ascertain the facts as to how frames of certain assumed proportions will naturally behave and what kind of modification is necessary to compel them to behave in a certain predetermined manner. The question of economy consistent with strength and stiffness is a compelling one, of course, but the relations among these three essential factors were not considered as a part of this study.

With regard to Conclusion (1) of the paper, the writers have subsequently shown that, for any bent of the type under consideration, for which both girders and columns are of the proportions required for vertical floor loads (assuming girders to be non-continuous and neglecting effect of column distortion from direct wind stress), the following relations are generally true, regardless of the number of columns, the number and height of stories, the length of girders, and the values of horizontal forces at the different floors:

(1) Girder moments are equal at both ends of any girder;
(2) Girder moments are proportional to their length;
(3) Girder shears are equal in all girders of any floor; and
(4) Direct stresses and vertical reactions for all interior columns are equal to zero.

These conditions were practically fulfilled for two assumed cases, applying Equations (10) to (17), inclusive, in the discussion by Mr. Mensch.[25b]

Thus, assuming $L_0 = 1$; $L_i = 2$; and $h = 1$, Equations (12) to (16), inclusive, become:

$$M_1 = \frac{4\,P}{29} \qquad \text{(41a)}$$

$$M_2 = \frac{3.5\,P}{29} \qquad \text{(41b)}$$

$$M'_2 = \frac{7\,P}{29} \qquad \text{(41c)}$$

$$R_A = \frac{7.5\,P}{29} \qquad \text{(41d)}$$

and,

$$R_B = -\frac{0.5\,P}{29} \qquad \text{(41e)}$$

Furthermore, assuming $L_0 = 3$; $L_i = 1$; and $h = 1$, the corresponding equations are:

$$M_1 = \frac{19\,P}{94} \qquad \text{(42a)}$$

[25b] Correction for *Transactions:* Equation (10): Change n to N in the term, $\frac{6}{1+n}$. In Equation (25), change M' to M'_2; and, on page 621, change Line 7 to read "$= -0.0588\, R_A$, against $-0.15\, R_A$ and $-0.19\, R_A$ from Models A-1 and A-1a."

$$M_2 = \frac{21\,P}{94} \quad \text{.......................(42b)}$$

$$M'_2 = \frac{7\,P}{94} \quad \text{.......................(42c)}$$

$$R_A = \frac{13.3\,P}{94} \quad \text{.......................(42d)}$$

and,

$$R_B = \frac{0.7\,P}{94} \quad \text{.......................(42e)}$$

Relation 4 was also practically fulfilled by the second model test reported by Mr. Morrison, and by Case *A*-1*c* of the paper, although in proportioning the frame for this latter case the writers did not have in mind the fact that the areas of the columns were proportional to those required for vertical floor loads. In these two cases the moments of inertia of columns were not in proportion to their areas, but the resultant reaction ratio in each case was approximately zero, thus practically agreeing with the portal theory.

The probability of such relations as those in Equations (41) and (42) was indicated by experimental results and recorded in the Fifth Progress Report of Sub-Committee No. 31 of the Structural Division, on Wind-Bracing in Steel Buildings[26], the proof being given in a discussion of the report.[27]

These relations are in exact agreement with the fundamental portal theory. It is thus interesting to discover that this theory has the backing of mathematical proof in the case of wind action on bents proportioned for vertical floor loads in the manner described in the aforementioned report.[26] The economic aspects of this conclusion would seem to be of considerable importance.

It is further of interest to note that if girders are assumed to act continuously, the interior column loads will be relatively increased and the other loads decreased, but an allowance for weight of outer walls will increase outer column loads and thus tend to offset the continuous girder effect. Under these conditions the relation between column loads across the bent will tend to approximate that for vertical floor loads alone with non-continuous girders.

[26] *Proceedings*, Am. Soc. C. E., March, 1936, pp. 410–411.

[27] See p. 1496.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

MODERN CONCEPTIONS OF THE MECHANICS OF FLUID TURBULENCE

Discussion

BY MESSRS. WARREN E. WILSON, AND THEODOR VON KÁRMÁN

WARREN E. WILSON,[43] JUN. AM. SOC. C. E. (by letter).[43a]—In presenting a useful summary of many of the modern developments of fluid mechanics the author has performed a valuable service to the Engineering Profession.

In the analysis of data on flow in open channels the Reynolds' number has been used to some extent and is expressed thus,

$$\mathsf{R} = \frac{V\,R}{\nu} \quad \text{.........................(91)}$$

in which the nomenclature is that of the paper. The author quite rightly questions the use of the hydraulic radius in this case.

The success which has been reported in such usage may be due in part at least to the fact that most experimental work on open channels covers a comparatively limited depth range. Furthermore, most experiments have been conducted with a small depth-width ratio, thereby giving a flow which approximates that in an open channel of infinite width.

Lindquist has suggested[44] plotting the dimensionless quantities, $\frac{V^2}{R\,g}$ and $\frac{V\,R}{\nu}$, as ordinates and abscissas, respectively, on logarithmic paper. The results obtained for several sets of experimental data, notably those of Darcy and Bazin, indicate an equation of the form:

$$\frac{V^2}{R\,g} = C\left(\frac{V\,R}{\nu}\right)^n \quad \text{.......................(92)}$$

However, if one were to plot a series in which the depth-width ratio varied over a range, such as 0.1 to 1.0, it seems entirely probable that the foregoing simple relation (Equation (92)) would no longer hold.

NOTE.—The paper by Hunter Rouse, Assoc. M. Am. Soc. C. E., was published in January, 1936, *Proceedings*. Discussion on the paper has appeared in *Proceedings*, as follows: April, 1936, by Chesley J. Posey, Jun. Am. Soc. C. E.; May, 1936, by Messrs. S. Franz Yasines, Benjamin Miller, and Ralph W. Powell; and August, 1936, by Joe W. Johnson, Jun Am. Soc. C. E.

[43] Instructor, Civ. Eng., South Dakota State School of Mines, Rapid City, S. Dak.

[43a] Received by the Secretary September 29, 1936.

[44] "Hydraulic Laboratory Practice", by the late John R. Freeman, Past-President and Hon. M. Am. Soc. C. E., p. 819.

With small values of the depth-width ratio, the problem involves wide channels approximately similar, geometrically. With successive cross-sections of a stream in a series covering widely differing values of the depth-width ratio there are channels that do not even approximate a similar cross-section. In a wide channel the bottom shear is by far the predominant retarding force. In a square cross-section the effect of side-wall shear is of the same order of magnitude as that of bottom shear.

For wide channels the ratio of unit shear on the bottom to that on the sides may be nearly the same throughout a series of experiments. Such is not the case if the depth-width ratio is increased to any great extent, thus leaving the range of wide channels.

It is impossible, therefore, to express the unit shearing stress at the boundary as a simple function of the velocity and, until a completely satisfactory theory of turbulent flow is available, an experimental approach to the problem seems in order. A determination of the distribution of shearing stress in the boundary surface as a function of the depth-width ratio would provide useful information for a further analysis of the flow in open channels.

THEODOR VON KÁRMÁN,[45] M. AM. SOC. C. E. (by letter).[45a]—It appears that this paper has two main objectives: First, to present the results of a series of experimental and theoretical investigations on the mechanism of turbulent flow to practical engineers; and, second, to suggest the use of rational formulas resulting from those investigations in design problems.

As far as the first objective is concerned, even the most practical engineer will agree that a better understanding of the underlying phenomena is useful also for the practical work. The only question is, whether in some cases total ignorance of some facts is not better than half-knowledge. Authorities disagree, however, on the second point. The writer has often been asked: "Are the new formulas free from any empirical element—that is, empirical assumptions and empirical constants? Why not use purely empirical formulas which have been carefully adjusted to the real conditions by experimental determination of the coefficients involved?" From a very competent source the writer learns that the formulas used for fluid resistance in actual practice are so exact that predictions agree with final results within 3 to 5%, whereas, unavoidable changes in the condition of the structures cause even larger differences.

Assuming that this statement is correct, nevertheless it appears to the writer that the rational formulas have at least three strong points:

First.—The rational formula is always dimensionally correct. This means that, in so far as the underlying assumptions are satisfied, it will give correct results independent of scale, absolute magnitude of speed, etc. Quite a few of the empirical formulas used in hydraulics are dimensionally incorrect. It seems that their success is based on the fact that experienced engineers use them only within a relatively narrow range.

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[45] Director, Daniel Guggenheim Aeronautical Laboratories, California Inst. of Technology, Pasadena, Calif.

[45a] Received by the Secretary October 30, 1936

Second.—Empirical formulas are good for interpolation, whereas rational formulas offer a fair chance for extrapolation. There is no reason why an empirical formula will give correct results beyond the limits of actual experiments. If, for instance, the friction coefficient, which is supposed to be a function of Reynolds number, was found, within a certain range, proportional to a certain power of that number, there is no justification to continue this law beyond the maximum Reynolds number used in the experiments. It is a common experience that almost every empirical function plotted in a suitable chosen scale on a logarithmic paper appears as a straight line within a small range. The theoretical formula for fluid friction in a smooth pipe suggested by the writer and given in the paper is deduced under the assumption that the influence of viscosity is negligible, except within a small range near the walls. Hence, it is probable that the formula will agree with the facts more and more, the larger the Reynolds number corresponding to the actual construction.

Third.—The theoretical formulas facilitate the transfer of results from one field to other related fields of research. Fluid friction and heat transfer in fluids are good examples for this point. Both phenomena are affected by the same mechanism of the turbulent flow. The problem of heat transfer is much more complex than the problem of friction. For instance, the friction coefficients for a smooth plate is a function of one parameter (Reynolds number) only, whereas the heat transfer depends on two parameters—namely, in addition to the Reynolds number, the ratio between the heat conduction coefficient and the product of heat capacity and viscosity. Hence, the flow conditions in oil and water will be similar if the corresponding Reynolds numbers are the same; but the heat flow will be different due to the relatively much smaller heat conduction coefficient of the oil. For this reason the heat transfer problem appears confusingly complicated from the standpoint of pure empirical research, and very elaborate series of research work in the past have yielded only information relating to a small section of the entire problem.

The use of the results of the turbulence theory determines the general shape of the heat transfer formula, indicates the manner in which the Reynolds number will enter into the formula, and reduces the problem to the determination of one single function of one variable which is the ratio involving the aforementioned physical constant. In a similar manner the writer believes that the turbulence theory and the use of its results will facilitate the systematic solution of many other complex problems, as, for instance, that of silt transportation, which is now the center of interest.

For the reasons presented herein, papers such as that by Mr. Rouse are extremely useful, also, from a merely practical standpoint. The difficulty is to decide at which stage a certain theory is ripe enough to be injected into the pulsating veins of engineering routine practice. Handicapped, perhaps, by the fact that he had some opportunity to work on the theory presented, the writer has been somewhat cautious about making such a proposition. However, the results obtained in aeronautical practice, with the corresponding skin-friction formulas, are so satisfactory that it now seems to the writer that the introduction of rational formulas in hydraulics should be encouraged.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

ANALYSIS OF VIERENDEEL TRUSSES

Discussion

BY MESSRS. L. J. MENSCH, A. A. EREMIN, LEON BLOG, A. W. FISCHER, AND L. C. MAUGH

L. J. MENSCH,[13] M. AM. SOC. C. E. (by letter).[13a]—From his splendid analysis of open-web trusses and bents Professor Vierendeel developed practical design formulas which relieve the structural engineer from the necessity of deep thinking when designing such structures. Why then has this practical method been ignored by American engineers, and by many European engineers? Professor Young's paper gives an answer to this question. The analysis is difficult to follow; there is a bewildering number of unknowns shown in Fig. 2, and in Fig. 5; in the absence of adequate explanations the directions of the unknown moments and shears are difficult to trace; and the reason for Δx being zero is rather difficult to find. The most serious obstacle to the proper understanding of the analysis is the omission of diagrams showing the deformation of the truss members under the assumed loading. The fact that Professor Young begins with a rather difficult case is another impediment.

The object of this discussion is to make it easy for the young engineer to grasp Professor Vierendeel's thoughtful conception by describing step by step how the simple open-web truss with symmetrical parallel chords deforms. Fig. 11(a) shows the typical deformation of every member of such a truss or bent, and the reader who has followed the latest discussions on wind-bracing bents will be surprised by the coincidence of the deformations. On account of the symmetry it is not difficult to see that the point of contraflexure of the web members must be in the center of the spans. As a rule, the points of contraflexure of the chords will not be in the center of the panels, but will be near the support or free ends, farther from the end than the center line of the panel, and will be found farther to the left in the panels near the center line of the truss or near a fixed end, as will be shown in detail subsequently.

Fig. 11(b) shows part of a half truss cut along the horizontal center line. In order to re-establish equilibrium it is necessary to apply, at each point of

NOTE.—The paper by Dana Young, Assoc. M. Am. Soc. C. E., was published in August, 1936, *Proceedings*. This discussion is printed in *Proceedings*, in order that the views expressed may be brought before all members for further discussion of the paper.

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[13a] Received by the Secretary September 3, 1936.

contraflexure of the web members, one-half the force acting at each panel point, this force being in the direction of the web member; and in order to prevent confusion in the signs of these forces, all forces, P, have been assumed to act

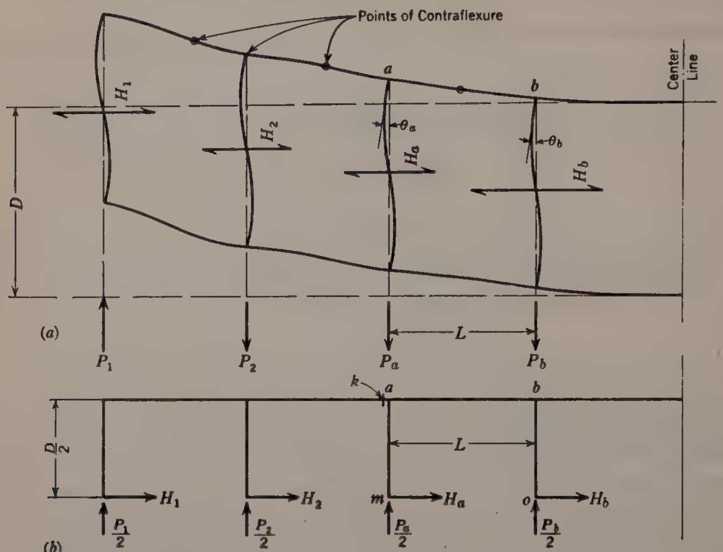

FIG. 11

in the same direction as in a wind-bracing bent. At the same points of contraflexure in the web members it is also necessary to apply the shear forces, H_1, H_2, H_a, etc., in a direction such that they will cause a curvature of the web member, as shown in Fig. 11(a).

By inspecting Fig. 11(b) one can readily see that all longitudinal forces and bending moments in each member may be found, after the statically unknown forces, H_1, H_2, etc., are known. The assumption that the longitudinal deformations of the chord members are so small that they may be neglected safely, leads peremptorily to the conclusion that the distance between the points of contraflexure of the web members is the same after, as before, the deformation of the truss.

At a section, k, an infinitesimal distance from Point a in Fig. 11(b), the left part of the truss or bent will exert the following forces and moments:

A longitudinal compression $= H_1 + H_2 = \sum_1^{a-1} H$; a shear in an upward direction $= \frac{P_1}{2} + \frac{P_2}{2} = \frac{1}{2} \sum_1^{a-1} P$; a right-hand moment from $\frac{1}{2} \sum_1^{a-1} P$ (which in the author's notation is designated, $\frac{1}{2} M_a$); and a left-hand moment from $\sum_1^{a-1} H$, or $\frac{1}{2} D \sum_1^{a-1} H$.

Fig. 12 shows an exaggerated picture of the deformed half-panel, *a-b*. If the deformation of the chords may be neglected, one may assume that $m\text{-}o = m'\text{-}o'$. The lines, $b'\text{-}o''$ and $a'\text{-}m''$, are drawn at right angles to the tangents at the chords at these points, and $a'\text{-}m'''$ was drawn parallel to $b'\text{-}o''$.

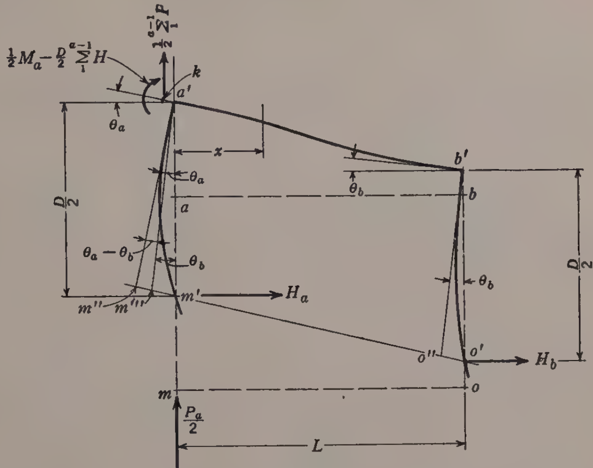

FIG. 12

Angles θ_a and θ_b are the rotations of the chord at a' and b', respectively, and $\theta_a - \theta_b$ is the change of angle of these rotations due to the moments acting in the chord, *a-b*, which may be found by the well-known formula:

$$\theta_a - \theta_b = \frac{1}{E\,I_c} \int_a^b M\,dx \qquad (92)$$

in which M is the moment acting at any point distant x from Point *a*.

The expression for M is found by adding to the moment acting at Section k: (*a*) The moment from the shear force acting at k, or $\frac{x}{2} \sum_1^{a-1} P$; (*b*) the moment from $\frac{1}{2} P_a$, or $\frac{1}{2} P_a\,x$; and (*c*) the moment from H_a, or $\frac{1}{2} D\,H_a$, which acts in the opposite direction. Moments (*a*) and (*b*) may be contracted to $\frac{x}{2} \sum_1^{a} P$, so that:

$$M = \frac{1}{2} M_a - \frac{1}{2} D \sum_1^{a-1} H + \frac{x}{2} \sum_1^{a} P - \frac{1}{2} D\,H_a$$

$$= \frac{1}{2} M_a - \frac{1}{2} D \sum_1^{a} H + \frac{x}{2} \sum_1^{a} P \qquad (93)$$

and,

$$\theta_A - \theta_B = \frac{1}{E\,I_c} \int_0^L \left(\frac{1}{2} M_a - \frac{1}{2} \sum_1^{a} H + \frac{x}{2} \sum_1^{a} P \right) dx \qquad (94)$$

In the author's notation, $\sum_{1}^{a} P = V_{ab}$, and,

$$\theta_A - \theta_B = \frac{1}{E\,I_c}\left(\frac{L}{2}\,M_a - \frac{D\,L}{2}\sum_{1}^{a} H + \frac{1}{2}\,V_{ab}\,\frac{L^2}{2}\right) \ldots\ldots (95)$$

For the movements, o'-o'' and m'-m'', the author gives the correct values in Equations (2), and the elastic equation which enables one to find the indeterminate values of the horizontal shear forces is obtained by,

$$m'\text{-}m'' = m''\text{-}m''' + m'''\text{-}m' = \frac{1}{2}\,D\,(\theta_A - \theta_B) + o'\text{-}o'' \ldots\ldots (96)$$

o'-o'' being equal to m'-m''' by inspection of Fig. 12. Substituting Equations (2) and (95) in Equation (96):

$$\frac{H_a\,D^3}{24\,E\,I_a} = \frac{H_b\,D^3}{24\,E\,I_b} + \frac{1}{2}\,\frac{D}{E\,I_c}\left(\frac{1}{2}\,M_a\,L - \frac{1}{2}\,D\,L\sum_{1}^{a} H + \frac{1}{2}\,L^2\,\frac{1}{2}\,V_{ab}\right) \ldots (97)$$

In the rare case in which $I_a = I_b = I_c$, Equation (97) reduces to:

$$H_b = H_a + 6\,\frac{L}{D}\sum_{1}^{a} H - \frac{6\,L}{D^2}\,M_a - 3\,\frac{L^2}{D^2}\,V_{ab} \ldots\ldots\ldots\ldots (98)$$

which is exactly equivalent to Equation (11), since $M_b = M_a + L\,V_{ab}$.

Professor Vierendeel simplified Equation (98) by introducing the moment of the exterior forces about a point exactly midway between a and b, which he called $M^b{}_a$, and which equals $M_a + \frac{1}{2}\,L\,V_{ab}$, and Equations (11) and (98) reduce to:

$$H_b = H_a + 6\,\frac{L}{D}\sum_{1}^{a} H - \frac{6\,L\,M^b{}_a}{D^2} \ldots\ldots\ldots\ldots\ldots\ldots (99)$$

For the more practical case in which the moments of inertia of the various members are different as long as the I-value of the chords in each particular panel are alike due to the assumption of symmetry, the writer has used the equation:

$$\frac{I_a}{I_b}\,H_b = H_a + 6\,n\sum_{1}^{a} H - 6\,\frac{n}{D}\,M^b{}_a \ldots\ldots\ldots\ldots\ldots (100)$$

which he derived from Equation (97) by introducing the relative stiffness of the web member to the chord in Panel ab:

$$n = \frac{I_a}{D} \div \frac{I_c}{L} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (101)$$

The writer remembers that, when he first studied Professor Vierendeel's analysis, he had some difficulties with the statement that H is to be set to zero at the base of a fixed bent, even where no shear member exists. Fig. 12

shows that, in this particular case, Point o'' returns to Point o'; Point m''' returns to Point m'; and Equation (96) must be replaced by:

$$m'\text{-}m'' = \frac{1}{2} D \, \theta_a \quad \text{.....................(102)}$$

The first term is, again, $\frac{H_a D^3}{24 \, E I_a}$, and the second term is given again by Equation (95). Therefore, it is clear that the term of H_b must really be set equal to zero in Equations (11), (97), (98), (99), etc., when it is used in the lowest panel of a fixed bent even where no web member exists.

Another difficulty encountered by students when they first try to apply this analysis occurs at the base when the ends are hinged; the solution is easy as the sum of all H-values must be equal to the reaction, which, in this particular case, is given by $\frac{M_b}{D}$. Often, there are solid walls as web members at the support of open web trusses. This case, also, is easily solved by omitting the first term in Equation (97), I_a being many hundred times larger than I_b.

American engineers should be vitally interested in Professor Vierendeel's analysis as it is the quickest method known for finding the moments and shears in the top and bottom stories of a wind-bracing bent, in which the points of contraflexure of the columns may not be near the middle of the story height or may be entirely absent, as shown in Fig. 11(*a*) in the panel near the center line of the truss. To illustrate, let Fig. 13 represent a six-story

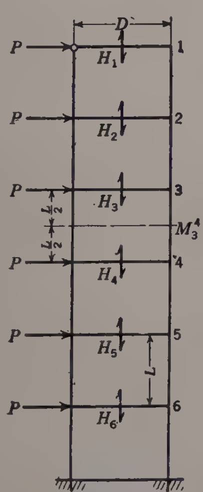

FIG. 13

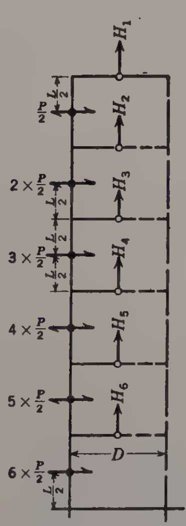

FIG. 14

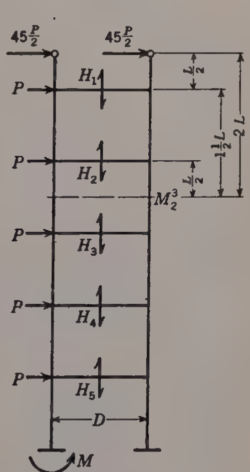

FIG. 15

wind-bracing bent. A force, P, is acting at each floor and the assumption is made that all columns have the same length, L, and the same moment of

inertia, I_c, and that all girders have the same moment of inertia, I_a. Equation (100) becomes,

$$H_b = H_a + 6\,n \sum_1^a H - 6\,\frac{n}{D}\,M^b{}_a \qquad (103)$$

When the relative stiffness of girders and columns is $n = \frac{1}{2}$, for example,

$$H_b = H_a + 3 \sum_1^a H - \frac{3}{D}\,M^b{}_a \qquad (104)$$

The values of $M^b{}_a$ for the various panels from the top down are easily found as $P\,L$ times 0.5, 2, 4.5, 8, 12.5, and 18, and one can write the elastic equation as follows:

$$H_2 = H_1 + 3\,H_1 - \frac{3\,P\,L}{2\,D} = 4\,H_1 - 1.5\,\frac{P\,L}{D} \qquad (105a)$$

and, $\sum_1^2 H = 5\,H_1 - 1.5\,\frac{P\,L}{D}$;

$$H_3 = H_2 + 3\left(5\,H_1 - 1.5\,\frac{P\,L}{D}\right) - \frac{6\,P\,L}{D} = 19\,H_1 - 12\,\frac{P\,L}{D} \qquad (105b)$$

and, $\sum_1^3 H = 24\,H_1 - 13.5\,\frac{P\,L}{D}$;

$$H_4 = H_3 + 3\left(24\,H_1 - 13.5\,\frac{P\,L}{D}\right) - 3 \times 4.5\,\frac{P\,L}{D} = 91\,H_1 - 66\,\frac{P\,L}{D} \qquad (105c)$$

and, $\sum_1^4 H = 115\,H_1 - 79.5\,\frac{P\,L}{D}$;

$$H_5 = H_4 + 3\left(115\,H_1 - 79.5\,\frac{P\,L}{D}\right) - 3 \times 8\,\frac{P\,L}{D} = 436\,H_1 - 328.5\,\frac{P\,L}{D} \qquad (105d)$$

and, $\sum_1^5 H = 551\,H_1 - 408\,\frac{P\,L}{D}$;

$$H_6 = H_5 + 3\left(551\,H_1 - 408\,\frac{P\,L}{D}\right) - 3 \times 12.5\,\frac{P\,L}{D} = 2\,089\,H_1 - 1\,590\,\frac{P\,L}{D} \qquad (105e)$$

and, $\sum_1^6 H = 2\,640\,H_1 - 1\,998\,\frac{P\,L}{D}$; and,

$$H_7 = 0 - H_6 + 3\left(2\,640\,H_1 - 1\,998\,\frac{P\,L}{D}\right) - 3 \times 18\,\frac{P\,L}{D}$$

$$= 10\,009\,H_1 - 7\,638\,\frac{P\,L}{D} \qquad (105f)$$

Whence,

$$H_1 = 7\,638 \frac{P\,L}{10\,009\,D} = 0.76311 \frac{P\,L}{D} \quad \ldots\ldots\ldots\ldots\ldots (106)$$

Substituting Equation (106) in Equations (105*a*) to (105*f*), the other indeterminate shear values, H_2, H_3, etc., may be found quickly.

Values of $H \left(\text{to be multiplied by } \frac{P\,L}{D} \right)$ in a six-story bent, for ratios of $n = \frac{1}{10}, \frac{1}{6}, \frac{1}{2}$, and 1, are given in Table 1.

TABLE 1.—VALUES OF H IN TERMS OF $\frac{P\,L}{D}$

Relative stiffness, n	H_1	H_2	H_3	H_4	H_5	H_6
$\frac{1}{10}$	1.282	1.75	2.38	2.91	3.08	2.42
$\frac{1}{6}$	1.09	1.68	2.45	3.17	3.56	3.01
$\frac{1}{2}$	0.76311	1.55	2.50	3.45	4.20	4.14
1	0.64545	1.52	2.50	3.48	4.34	4.20
Common theory	0.50	1.50	2.50	3.50	4.50	5.50

It will be interesting to compare the shear values found by Professor Vierendeel's analysis with those computed by the common theory which is based upon the assumption that the point of contraflexure of the columns is at the mid-story height. Fig. 14 illustrates this assumption and by taking moments about the juncture of column and girder the following statical equations may be written, from the top downward:

$$\frac{1}{2} P \times \frac{1}{2} L = H_1 \times \frac{1}{2} D; \text{ or, } H_1 = \frac{1}{2} \frac{P\,L}{D}$$

$$\frac{1}{2} P \times \frac{1}{2} L + 2 \times \frac{1}{2} P \times \frac{1}{2} L = H_2 \times \frac{1}{2} D; \text{ or, } H_2 = 1.5 \frac{P\,L}{D}$$

$$2 \times \frac{1}{2} P \times \frac{1}{2} L + 3 \times \frac{1}{2} P \times \frac{1}{2} L = H_3 \times \frac{1}{2} D; \text{ or, } H_3 = 2.5 \frac{P\,L}{D}$$

$$3 \times \frac{1}{2} P \times \frac{1}{2} L + 4 \times \frac{1}{2} P \times \frac{1}{2} L = H_4 \times \frac{1}{2} D; \text{ or, } H_4 = 3.5 \frac{P\,L}{D}$$

$$4 \times \frac{1}{2} P \times \frac{1}{2} L + 5 \times \frac{1}{2} P \times \frac{1}{2} L = H_5 \times \frac{1}{2} D; \text{ or, } H_5 = 4.5 \frac{P\,L}{D}$$

and,

$$5 \times \frac{1}{2} P \times \frac{1}{2} L = 6 \times \frac{1}{2} P \times \frac{1}{2} L = H_6 \times \frac{1}{2} D; \text{ or, } H_6 = 5.5 \frac{P\,L}{D}$$

Comparing the values given in Table 1 it will be noted that the shear, H_1, for the top girder is entirely too small as derived by the common theory, whereas H_2 computed by the common theory nearly agrees with the values found by this analysis for $n = \frac{1}{2}$ and 1 and is only 16% at variance for $n = \frac{1}{10}$.

For H_3, which is only two panels down from the top (a point of singularity), the agreement is quite remarkably close. The great variation of stiffness has scarcely any influence on the value of H_3 and it is quite permissible to compute it by the common theory. Another point of singularity is found at the bottom of the structure where the discrepancies in the shear values of H_6 are as great as for H_1 at the top. The irregularity at the bottom extends for three panels in the case of $n = \frac{1}{10}$ and $n = \frac{1}{6}$, and for two panels or even only one panel, in the case of $n = \frac{1}{2}$ and $n = 1$.

It is interesting to compute the moment at the base of the columns for the case, say, of $n = \frac{1}{10}$. The sum of all H-values is found from Table 1 to be $13.83 \frac{P\,L}{D}$, whereas the moment of one-half the exterior forces about the base is 10.5 PL; that is, the moment at the base:

$$M = 10.5\;P\,L - \frac{1}{2}\;D \times 13.83\;\frac{P\,L}{D} = 3.59\;P\,L \ldots\ldots\ldots (107)$$

which is very much larger than the value of $6 \times \frac{1}{2}\;P \times \frac{1}{2}\;L$ found by inspection of Fig. 14 by using the common theory.

The shear at the base being only 3 P, it requires a leverage of more than the story height to produce a moment of 3.59 $P\;L$. Therefore, there is no point of contraflexure in the lowest story for this particular case ($n = \frac{1}{10}$).

From the foregoing considerations it follows that, in a tall building bent with many stories, the shears in the girders (except, possibly, two stories at the top and two to six stories at the bottom) may be computed by the common theory; that is, by assuming ideal hinges in the columns at their mid-story heights. For the computation of H_1 and H_2 the writer uses the following short-cut: Write the elastic equations for H_2 and H_3 by using Equation (103); then compute H_3 by the common theory; and equate the two values. For example, in the case of the six-story frame, shown in Fig. 13, for $n = \frac{1}{2}$, Equation (105*b*), $H_3 = 19\;H_1 - 12\;\frac{P\,L}{D}$, which must be equal to the value of $H_3 = 2.5\;\frac{P\,L}{D}$ found by the common theory as given in Table 1. Therefore, $2.5\;\frac{P\,L}{D} = 19\;H_1 - 12\;\frac{P\,L}{D}$; and $H_1 = 0.763\;\frac{P\,L}{D}$, which is nearly the same as the value given in Table 1.

For computing the girder shears in the lowest stories of tall bents (say, of 50 stories), the following procedure will save considerable time: At each point of contraflexure of the columns, just above the fifth story (see Fig. 15), apply one-half the wind load $\left(\text{namely, } 45\;\frac{P}{2}\right)$, and then write the elastic equations of the type of Equation (103). Assuming that $n = \frac{1}{10}$, Equation (103) becomes:

$$H_b = H_a + 0.6 \sum_{1}^{a} H - 0.6\;M^b{}_a \ldots\ldots\ldots\ldots\ldots (108)$$

and the values of $M^b{}_a$ for the various stories from the fifth story down, are found as $P\ L$ times 45.5, 92, 139.5, 188, and 237.5, respectively, or,

$$H_2 = H_1 + 0.6\ H_1 - 0.6 \times 45.5\ \frac{P\ L}{D} = 1.6\ H_1 - 27.3\ \frac{P\ L}{D} \quad \text{(109a)}$$

$$H_3 = H_2 + 0.6\left(2.6\ H_1 - 27.3\ \frac{P\ L}{D}\right) - 0.6 \times 92\ \frac{P\ L}{D}$$

$$= 3.16\ H_1 - 98.88\ \frac{P\ L}{D} \quad \text{(109b)}$$

$$H_4 = H_3 + 0.6\left(5.76\ H_1 - 126.2\ \frac{P\ L}{D}\right) - 0.6 \times 139.5\ \frac{P\ L}{D}$$

$$= 6.616\ H_1 - 258.3\ \frac{P\ L}{D} \quad \text{(109c)}$$

$$H_5 = H_4 + 0.6\left(12.376\ H_1 - 384.5\ \frac{P\ L}{D}\right) - 0.6 \times 188\ \frac{P\ L}{D}$$

$$= 14.04\ H_1 - 601.6\ \frac{P\ L}{D} \quad \text{(109d)}$$

and,

$$0 = H_6 = H_5 + 0.6\left(26.42\ H_1 - 985.1\ \frac{P\ L}{D}\right) - 0.6 \times 237.5\ \frac{P\ L}{D}$$

$$= 29.9\ H_1 - 1\,335.2\ \frac{P\ L}{D} \quad \text{(109e)}$$

Hence,

$$H_1 = 1\,335.2\ \frac{P\ L}{29.9\ D} = 44.6\ \frac{P\ L}{D} \quad \text{(110)}$$

Equation (110) substituted in Equations (109) results in: $H_2 = 44.1\ \frac{P\ L}{D}$; $H_3 = 42.1\ \frac{P\ L}{D}$; $H_4 = 39\ \frac{P\ L}{D}$; and, $H_5 = 24.6\ \frac{P\ L}{D}$. By the common theory, the shear for the fifth floor girder is obtained by the equation, $H_1 \times \frac{1}{2}\ D = 22.5\ P \times \frac{1}{2}\ L + 23\ P \times \frac{1}{2}\ L$; or,

$$H_1 = 45.5\ \frac{P\ L}{D} \quad \text{(111)}$$

which varies only by 2% from the value in Equation (110).

The study of the two-column bent may be safely used in estimating the shear values in the girders of the upper two stories and those of the lowest two to six stories of a bent with three, four, or more columns. For this purpose an estimate of the column shears of the outside columns must be obtained by any valid method, such, for example, as the writer has indicated elsewhere.[14] Twice this shear must be considered as the load on the two-column

[14] *Journal*, Am. Concrete Inst., February, 1932; and *Proceedings*, Am. Soc. C. E., April, 1936, p. 615.

bent and the girder shears may be computed for the irrgular panels at the top and bottom, as shown previously herein.

The shears thus found in the girders will differ from those found by the common theory, and the shears in the corresponding girders of the inside bays may be changed in the same proportion, provided there is not too great a change in the value of the corresponding value of n. Some objections may be raised to the use of the formulas for a symmetrical two-column bent to the case of a bay cut from a regular wind-bracing bent where the two columns may not be of the same stiffness. The writer has studied, thoroughly, the two-column bent with columns of different stiffness (the unsymmetrical bent), and has found that, except for the top and bottom stories, the following relation holds:

$$\frac{X_1}{X_2} = \frac{(6 + n'')}{(6 + n')} \qquad (112)$$

in which X_1 is the shear at the left column; X_2 is the shear at the right column; n' is the relative stiffness between the girder and the left column; and n'' is the stiffness in relation to the right column. From Equation (112) one may observe at once that the shears in the columns will vary only a little when the values of n' and n'' are small, even where they differ from each other by as much as 100 per cent. For $n' = \frac{1}{8}$ and $n'' = \frac{1}{4}$ $\frac{X_1}{X_2} = 1.022$, from which it may be concluded that no great errors will be made in most cases of unsymmetrical frames when a corresponding symmetrical frame is analyzed first and the possible variation is studied later.

For end conditions the factor, 6, had better be replaced by 1 in Equation (112). If the author is correct in stating (see heading "Special Approximations") that Professor Vierendeel uses the assumption:

$$\alpha = \beta = \frac{I'_c L}{I''_c L_t} \qquad (113)$$

then the writer is convinced that Professor Vierendeel's students did not accept his analysis seriously or they would have analyzed many examples and would have found that Equation (113) "falls far from the mark"; although it does not seem to effect the results in Example 4 of the paper.

Professor Young's paper is an important contribution to American literature on structural engineering. However, it is no improvement on the 40-yr old method introduced by Professor Vierendeel. In fact, many of his practical simplifications have been omitted. The paper deserves the earnest discussion of the keenest talent in the United States because this analysis is susceptible of great simplification to the vast benefit of structural engineers, especially at this time when arithmetical methods are lauded as being preferable to a clear understanding and scientific processes.

Of course, Professor Young's analysis is not "exact" even in his own sense. He does not say whether clear spans or center-to-center spans should be used in the calculations and there is certainly a difference of at least 20% from this source alone, in most cases. There are considerable "roundings" at all the

junctures of web members and chords, and they may change the results 5 to 10 per cent. If to this is added the serious error made in all textbook rules for the design of members subject to combined loading (which are often 50% in error), it is quite probable that an investigator who tests a model to destruction which has been designed by Professor Young's "exact" method may find it 100% too strong and will declare Professor Young's method a "safe" guide.

Fig. 16 shows a 100 000-gal sprinkling tank on a 80-ft tower, built in 1911 for the Chicago City Railway Company, Chicago, Ill., and Fig. 17 shows one

FIG. 16—VIEW OF 100 000-GALLON SPRINKLING TANK ON 80-FOOT TOWER, CHICAGO, ILL.

FIG. 17—VIEW SHOWING ONE OF SEVERAL OPEN-BENT STRUCTURES, COAL-BIN OF 1 900-TON CAPACITY, GRANITE CITY, ILL.

of several open-bent structures (a coal-bin of 1 900-ton capacity, 142 ft high) built for the St. Louis Coke and Chemical Company, in 1919, at Granite City, Ill. The latter is probably the most unusual Vierendeel type of structure ever built. A large number of open-bent structures have been built in the United States.

A. A. EREMIN,[15] ASSOC. M. AM. SOC. C. E. (by letter).[15a]—Although little has been written on this subject in English, the distribution of stresses in Vierendeel trusses has been studied extensively abroad. An exact computation of these stresses is a forbidding task and, therefore, various simplified methods of computation have been developed. The advantage of a Vierendeel truss is that, for all practical purposes, stresses can be computed with reasonable accuracy for assumptions as to distribution.

The method described by the author is not simple, although it can easily be simplified. For example, in the case of a Vierendeel truss with parallel chords and constant moment of inertia, Equation (11) may be written:

$$H_{bw} = H_{aw} + 6\,\frac{L}{D}\sum_{1}^{a} H_w - 6\,\frac{L}{D^2}\left(M_b + \frac{1}{2}\,L\,V_{ab}\right) \quad \ldots\ldots (114)$$

The bending moment, M_{ab}, due to external forces carried by the truss, taken at the mid-length of Panel ab, may be written:

$$M_{ab} = M_b + \frac{1}{2}\,L\,V_{ab} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (115)$$

Substituting Equation (115) in the last term of Equation (114):

$$H_{bw} = H_{aw} + 6\,\frac{L}{D}\sum_{1}^{a} H_w - 6\,\frac{L}{D^2}\,M_{ab} \quad \ldots\ldots\ldots\ldots (116)$$

It is evident that Equation (116) requires less computation than Equation (11). It may also be noted that Equation (116) is exactly the same as that developed by Dr. F. Gebauer[16] prior to 1907.

Computation for this case may be simplified further by assuming that points of contraflexure in all members are at their mid-lengths, as shown in Fig. 18. In Fig. 19 the vertical member at Joint b is shown, with the forces

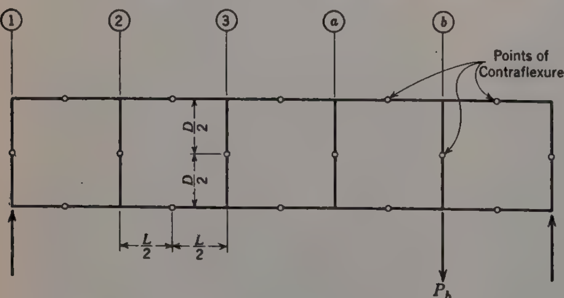

FIG. 18

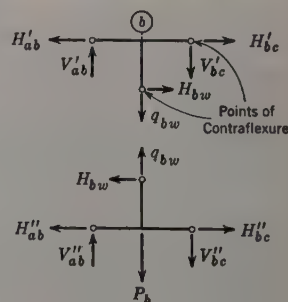

FIG. 19

acting at the points of contraflexure. Stresses indicated in these diagrams may be determined by statics. Using the author's notation, direct stresses in the chords are:

$$H'_{ab} = -\,H''_{ab} = \frac{M_a + M_b}{2\,D} \quad \ldots\ldots\ldots\ldots\ldots\ldots (117)$$

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[15] Assoc. Bridge Designing Engr., Div. of State Highways, Sacramento, Calif.
[15a] Received by the Secretary September 14, 1936.
[16] *Beton und Eisen*, 1907, p. 252.

The shear stresses, V'_{ab}, at the points of contraflexure in the chords at Panel *ab* are:

$$V'_{ab} = -V''_{ab} = \frac{1}{2} V_{ab} \quad \text{.................... (118)}$$

The shear stresses, H_{bw}, at the point of contraflexure in vertical members at Joint *b* are:

$$H_{bw} = H_{bc} + H_{ab} \quad \text{.................... (119)}$$

The direct stress, q_{bw}, in the vertical member at Joint *b* is,

$$q_{bw} = V_b - P_b \quad \text{.................... (120)}$$

The sum of the bending moments at the ends of members meeting at any joint is equal to zero; thus, for Joint *b* at the upper chord, an expression may be written (see Fig. 19):

$$V'_{ab} \frac{L}{2} + V'_{bc} \frac{L}{2} - H_{bw} \frac{D}{2} = 0 \quad \text{.................... (121)}$$

Equation (121) may also be used for checking the computations.

Referring to Example 1 of the paper, the shear stress at the end vertical member computed by means of Equation (117) is, $H_{1w} = 750$ lb; likewise, $H_{2w} = 1\,000$ lb. Therefore, it is evident that the shear stress at the end vertical member computed by means of Equation (117) differs slightly from that computed by Equation (11). The error increases toward mid-span, but will be less in Vierendeel trusses with a large number of panels. Furthermore, there is generally a greater factor of safety in verticals toward the mid-span, due to architectural requirements that vertical members be uniform.

If, in Vierendeel trusses with parallel chords, the moment of inertia of the upper chord is I_t, and the moment of inertia of the lower chord is I_b, the assumed distances of points of contraflexure in vertical members to top chords are:

$$a = D \frac{1}{1 + \dfrac{I_b}{I_t}} \quad \text{.................... (122)}$$

Equation (122) is similar to the formula for locating points of contraflexure developed by Professor R. Saliger.[17]

In the case of a Vierendeel truss with a curved top chord and straight bottom chord (see Fig. (5*a*)), a limitation of the author's method is that the true values of α and β can not be computed. In practice, the values of α and β differ slightly from unity. Furthermore, the probable error in the computation of the stresses (even with the help of modern mechanical calculating machines) is always greater than the difference between the true values of α and β and the values assumed. This is especially true in the case of Vierendeel trusses with a large number of panels.

[17] "Der Eisenbetonbau", 6 Aufl., Leipzig, 1933.

Equation (54) is reasonably simple and may be used for the general cases in practice. However, the computation of the moments, M_w, and the shear stresses, V', may be simplified. Assume that the points of contraflexure are at the middle of the vertical members and that the shear stress, H_w, is applied at those points. Then, the maximum bending moment, M_{aw}, in the vertical member at Joint a, is,

$$M_{aw} = H_{aw} \frac{D_a}{2} \qquad (123)$$

In Example 3 of the paper, the moments in the vertical members as computed by Equation (123) are: $M_{1w} = 601.96 \times 2 = 1\,203.92$; $M_{2w} = 366 \times 4 = 1\,464$; $M_{3w} = -10 \times 5 = -50$; $M_{4w} = -262 \times 4 = 1\,048$; and, $M_{5w} = -411 \times 2 = 822$.

The shear stress, V', may be computed from Equation (48a), assuming $\alpha = 1$; thus:

$$V'_{ab} = \frac{1}{2}\left(V_{ab} + \frac{d}{L}\sum_1^a H_w\right) \qquad (124)$$

Shear stresses computed by Equation (124) are: $V'_{12} = \frac{1}{2}(600 + 0.4 \times 600) = 420$; $V'_{23} = \frac{1}{2}[600 + 0.2 \times (600 - 369)] = 396.9$; $V'_{34} = -\frac{1}{2}(400) = -200$; $V'_{45} = -\frac{1}{2}[400 + 0.2\,(409 + 287)] = -269.6$; and $V'_{56} = -\frac{1}{2}[400 + 0.4\,(409)] = -281.8$.

It is evident that the moments and shear stresses determined by Equations (123) and (124), respectively, differ slightly from those computed by Equations (75) and (77) of the paper. The labor involved, however, is considerably reduced. The author is to be congratulated for his valuable and interesting contribution.

Leon Blog,[18] Assoc. M. Am. Soc. C. E. (by letter).[18a]—Formulas for analyzing several of the more familiar types of Vierendeel trusses are presented in this paper. As the largest part of the paper is devoted to the analysis and solution of problems dealing with the case of an inclined upper chord, the writer has confined his discussion to that phase.

Whether one will prefer to use the formulas presented by the author or the approximate equations derived by Professor Vierendeel, depends upon the exigencies of one's practice. Some engineers prefer to use formulas approximate to a known degree of accuracy and affording a speedy solution. They know full well that a liberal correction must be made to compensate for causes which make the theoretical stresses unattainable.

The main difference between the analyses of Professors Vierendeel and Young is that the former develops accurate formulas at the beginning of his study, demonstrates that they are too unwieldy for speedy solution, and, after making certain assumptions which carry conviction, evolves formulas which, although admittedly approximate, have a simple nomenclature, are easy to

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[18a] Received by the Secretary October 1, 1936.

apply, and, above all, are direct. No "cut-and-try" process is required. Professor Young evolves a formula for horizontal shear in the vertical members which is theoretically exact according to the postulates stipulated; but it is difficult to apply because it is not simple. It is expressed in terms of unknowns. It can only be made simple by assuming certain convenient relations between the moments of inertia of the truss members, the fulfillment of which, in practice, would be purely fortuitous. How often will ϕ equal 1? When it does not, the various moments of inertia remain in Equation (53) and the fractions containing α and β must be evaluated before that formula can be solved. The difficulty of evaluating α is indicated subsequently under the heading "Comparison of Formulas."

Using Professor Vierendeel's equations[19] the writer has made a check of the solution of Example 3 which the author solved by use of the formulas of the paper. The results of this solution are compared with that of the author in Fig. 20. All the forces are in pounds and the moments are in foot-pounds.

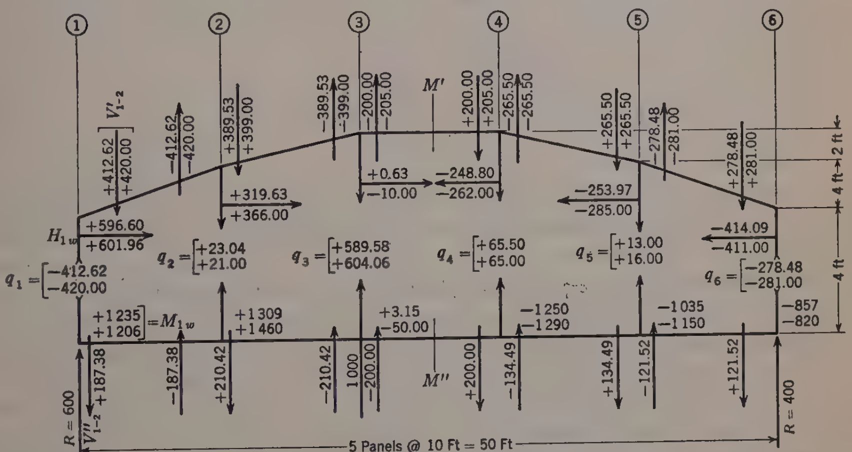

FIG. 20

Shear in the lower part of the vertical members balances the shears that are shown, and the writer's values are placed above, and to the left of, those determined by the author.

In the writer's opinion, the author did not depart essentially from the method of analysis used by Professor Vierendeel for the inclined upper chord truss. He arrived at different formulas by using the tool of least work to evaluate the relative displacements of the junctions of the verticals with the chords for two successive verticals. Professor Vierendeel based his method upon the linear and angular displacements of a point in the axis of an arch of flat curvature. He applied the formulas for these displacements to find the relative displacements of two successive verticals above a plane passed through them between the chords, and equated the displacements for the same points in the verticals below the plane. The result was an exact expression for

[19] "Cours de Stabilite des Constructions", Tome IV, by Arthur Vierendeel, Louvain, 1920, Equation (16); Equation (3) p. 194; Equation (4) p. 183; and Equations (8) and (9), p. 188.

the horizontal shear in any vertical which the author deemed to be too complex for easy and speedy solution. Making use of the relation which he had previously proved, that $\frac{M'}{M''} = \frac{I'\,dx}{I''\,ds}$, in which M' and I' are the internal bending moment and the moment of inertia at any section of the upper chord and M'' and I'' refer to a point in the lower chord in that same section, and since $\frac{dx}{ds}$ is the cosine of the angle of slope of the panel in which the section is taken, then, letting $\frac{I'\,dx}{I''\,ds} = \beta$, $M' = \beta\,M''$; that is, either chord moment is known when the other is known. This is the fundamental concept which underlies the derivations of formulas for the chord shears and horizontal shears through the verticals evolved by Professor Vierendeel. It is based upon the tenable assumption that the length of the verticals remains constant. This concept also underlies the location of the point of inflection of the vertical measured up to a distance, y, from the lower chord. When the curvature of the upper chord is not too sharp:

$$y = \frac{D}{1 + \frac{\beta + \beta_1}{2}} \qquad (125)$$

in which D[20] is the height of the vertical and β and β_1 are ratios of chord moments in the panels adjacent to the vertical considered. The true expression for the point of inflection is a function of the chord moments which cannot be determined before the horizontal shears in the verticals have been found. Another simplifying assumption that the upper chord has a finite area but an infinitely small moment of inertia, reduces the denominator of the value of y, as stated, to unity; that is, the point of inflection lies in the upper chord. If the upper chord at the vertical has no stiffness it can take no moment so that β and β_1 equal zero. This assumption may affect the value of the horizontal shear in a vertical as much as 5.5 per cent.[21] Based upon the simplifying assumptions made, and proceeding from the exact analysis for the horizontal shear in a vertical, Professor Vierendeel evolved the following formula:[22]

$$H_{r+1} = \frac{H_r\,D^2_r\,(3\,D_{r+1} - D_r)}{2\,D^3_r + 1} + \frac{L\,(3\,D_r\,D_{r+1} + d^2)}{D^3_{r+1}} \sum_1^r H - \frac{3\,L\,(D_r + D_{r+1})\,M_{r+1}}{2\,D^3_{r+1}} + \frac{L^2\,(2\,D_r + D_{r+1})\,V_r}{2\,D^3_{r+1}} \qquad (126)$$

The subscript, r, denotes the left-hand panel point whereas $r + 1$ denotes the one immediately to the right; H denotes the horizontal shear in the vertical; D, its height; d, the difference in heights of two successive verticals; L, the panel length; and, M and V, the external bending moment and shear, respec-

[20] "Cours de Stabilité des Constructions", Tome IV, by Arthur Vierendeel, Louvain, 1920, p. 186.

[21] *Loc. cit.*, p. 179.

[22] *Loc. cit.*, Equation (16), p. 194.

tively. Note the absence of all symbols except dimensions or forces in Equation (126) and the freedom from terms which themselves involve unknowns. Regardless of the value of H_r, $\sum_1^r H$, M_{r+1} or V_r, their coefficients are always the same and need be computed but once. Professor Vierendeel also reverses the position of the load in order to derive all the shears, as did the author.

Comparison of Formulas.—Vierendeel wrote a formula[21] for horizontal shear in vertical members as follows:

$$H_{bw} = \frac{D^2_a\, H_{aw}}{2\, D^3_b}\,(3\, D_b - D_a) + \frac{L}{D^3_b}\Big[(3\, D_b\, D_a) + (D_b - D_a)^2\Big] \sum_1^a H_w$$
$$- \frac{3\, L\, (D_b + D_a)}{2\, D^3_b}\, M_b + \frac{L^2\, (D_b + 2\, D_a)}{2\, D^3_b}\, V_a \quad \ldots\ldots\ldots\ldots (127)$$

in which the notation is that of the paper. It is obvious that Equation (127) is more simple than Equation (53), involving fewer terms, and including only the linear dimensions of the truss and the known external bending moments and shears. All the assumptions are involved in the formula and, assumptions that $\phi = 1$, or that $I_a = I_b = I''_c = I'_c \frac{L}{L_t}$, such as the author makes to solve Equation (53), need not be made The difficulty about using Equation (53) is that H_{bw} involves unknown chord moments represented by α and β. These chord moments are expressed, in turn, in terms of themselves and of H_w as shown by reference to Equations (50*a*) and (50*b*). The ratio, β, is known from the properties of the truss, but H_w must still be found before the moments can be determined. This is a circuitous process. Professor Vierendeel suggested[23]:

$$M''_{ba} = -\frac{M_{ba}}{1 + \beta} + \frac{D_{ba} \sum_1^a H_w}{1 + \beta} \quad \ldots\ldots\ldots\ldots\ldots (128)$$

It is to be noted that Equation (128) differs from Equation (50*b*) in the sign convention and that $\beta = \frac{M'_{ab}}{M''_{ab}}$.

The author's formulas for chord shears, Equation (48*a*) and Equation (48*b*), are complex because they involve the upper and lower chord moments of two adjacent panels which must first be found from Equation (53) and Equation (50*b*). Such moments need not be found when using the Vierendeel formulas[24],

$$V'_{ab} = \frac{V_{ab}}{1 + \beta} + \frac{\frac{d}{L} \sum_1^a H_w}{1 + \beta} \quad \ldots\ldots\ldots\ldots\ldots (129)$$

[23] "Cours de Stabilite des Constructions", Tome IV, by Arthur Vierendeel, Louvain, 1920, Equation (9), p. 188.

[24] *Loc. cit.*, Equations (3) and (4), p. 183.

and,

$$V''_{ab} = \frac{V_{ab}}{1+\beta} - \frac{\frac{d}{L}\sum_{1}^{a} H_w}{1+\beta} \quad \ldots\ldots\ldots\ldots\ldots\ldots (130)$$

because $\frac{M'_{ab}}{M''_{ab}} = \beta = \frac{I'\,dx}{I''\,ds}$, which is a known constant and, when substituted for the easily found $\Sigma\, H_w$, readily yields V'_{ab} and V''_{ab}.

Discussion of Results in Fig. 20.—Fig. 20 shows a comparison of all the values found by the author with those found by the Vierendeel formulas. In addition, the writer computed the shears in the lower chord by the Vierendeel formula, Equation (130). The sum of the shears in both chords in any panel must equal the external shear at that point and yields a check on the underlying evaluations.

The maximum shear variation in the upper chord occurs in Panel 3–4 and is 2.50% less than the value given by the author. The maximum significant horizontal shear variation in any vertical occurs in q_2 and is 12.50% less than the author's value. The maximum variation in direct stress in any vertical occurs in q_5 and is 18.70% less than the author's value. The maximum significant moment variation in any vertical occurs in q_2 and is 10.30 per cent.

The term, significant, is used to exclude the values for horizontal shear and bending moment in q_3. For shear, the writer obtained + 0.63 as against the author's − 10.00. These values are both small and of opposite signs. In any practical design, they would be allowed a factor of error because of the proximity to zero which might be due to the sensitivity of both methods to the underlying assumptions. For instance, in the author's solution of Examples 3 and 4, H_{3w} each time equals − 10 lb despite his different relations between the moments of inertia. For the moments in q_3, the writer's value is + 3.15 and that of the author − 50.00 ft-lb. The ratio of these shears and moments is about 15.9 to 1. Except for the values for q_3, the agreement between the two solutions is close.

Under the heading, "Special Approximations", the author states that "Professor Vierendeel uses an approximate method for calculating the chord moments and shears[12] which is very rapid. Equations (48) and (50) are expressions for these functions." He should have made it clear that Equations (48) and (50) are his own expressions. The corresponding Vierendeel formulas are Equations (128), (129), and (130).

Referring to Conclusion (3) of the paper, the writer agrees that the solution for a truss with an inclined upper chord although proceeding from an exact formula, must, in the last analysis, be approximate. As to Conclusion (4), the formulas proposed cannot be as rapid as those of Professor Vierendeel because the simplifying assumptions required to be made with regard to moments of inertia, ϕ, and values of α to arrive at a simpler formula (Equation (54) from Equation (53)) will not, in general, correspond

[12] "Cours de Stabilite des Constructions", Tome IV, by Arthur Vierendeel, Louvain, 1920.

with the conditions of the problem. These difficulties are eliminated by Professor Vierendeel, who makes rational assumptions during the development of his final simple and direct formulas. The author's remarks in Conclusion (5) apply to the Vierendeel formulas, but the numerical effect of Assumption (3) is not immediately evident except for those happily selected cases, such as for $\phi = 1$, unless a solution is made by another method as a check. Appreciable errors can be made when the slopes of the upper chord vary considerably from panel to panel. Equation (53) contains too many unknowns for a rapid solution having any assured degree of accuracy. How much error is involved when ϕ does not $= 1$ and when α and β are assumed?

To those who are interested in the subject of Vierendeel trusses, the writer recommends reference to Professor Vierendeel's comprehensive treatment of trusses of various shapes and purposes as well as the design of panel joints.[12] An analysis of Vierendeel trusses by the combined Cross and Grinter methods was presented in an able paper before the International Association for Bridge and Structural Engineers.[25]

From the standpoint of rapid design, the writer does not consider the author's formulas as rapid, or as unfailingly accurate, as those presented by Professor Vierendeel.

A. W. FISCHER,[26] ESQ. (by letter).[26a]—The author has certainly contributed a valuable paper to the Engineering Profession on the analysis of the Vierendeel truss with either symmetrical or inclined upper chords. At first glance the theory seems rather long, but after it is studied and understood it is very simple (as are the final general equations) in application, if the theory from which they were developed is understood.

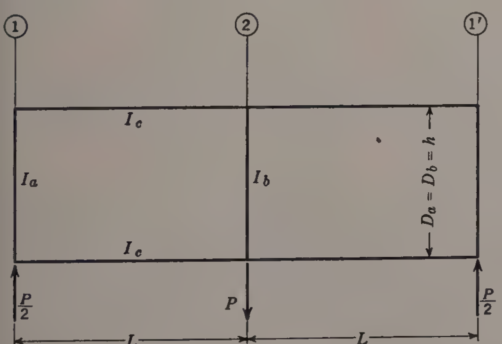

FIG. 21

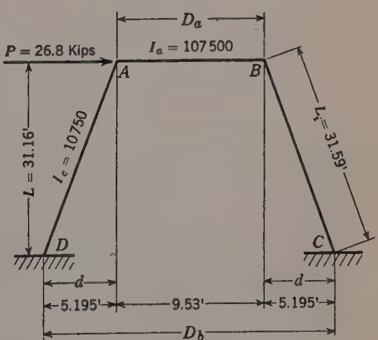

FIG. 22

Equation (10) is a general formula and solves symmetrical structures with either parallel or inclined chords in a short time.

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[25] Rept. of the International Assoc. for Bridge and Structural Engrs., Vol. 3, 1935, by L. C. Maugh, Assoc. M. Am. Soc. C. E.

[26] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[26a] Received by the Secretary October 3, 1936.

Example A.—For example, consider a symmetrical quadrangular bridge truss as shown in Fig. 21 which is a two-span Vierendeel truss, with parallel chords. Substituting in Equation (10):

$$H_{2w} = \frac{I_b\ H_{1w}}{I_a} + \frac{6\ L\ I_b\ H_{1w}}{h\ I_c} - \frac{6\ L\ I_b\,M_2}{h^2\,I_c} + \frac{3\ L^2\,I_b\ V_{12}}{h^2\,I_c} \quad \ldots.(131)$$

From the symmetry of the loading in this particular case, it is obvious that $H_{2w} = 0$; $M_2 = 0.5\,PL$; and $V_{12} = 0.5\,P$. Substituting in Equation (131):

$$\frac{H_{1w}}{I_a} + \frac{6\,L\,H_{1w}}{h\ I_c} - \frac{3\,L^2\,P}{h^2\,I_c} + \frac{1.5\,L^2\,P}{h^2\,I_c} = 0 \quad \ldots\ldots\ldots\ldots(132)$$

Reducing Equation (132):

$$H_{1w} = \frac{1.5\ P\,\phi\,\dfrac{L}{h}}{1 + 6\,\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(133)$$

in which $\phi = \dfrac{I_a\,L}{I_c\,h}$.

The moment at the intersection of the end vertical and the top chord,

$$M = 0.5\ H_{1w}\ h = \frac{0.75\ P\,\phi\,L}{1 + 6\,\phi} \quad \ldots\ldots\ldots\ldots\ldots\ldots(134)$$

which is the same value as that given elsewhere.[27] The moments at the other joints can now readily be computed by statics.

Example B.—As another example the writer will use the top story of the bent analyzed as Example 2 in the paper and will assume the bases fixed, so as to form a symmetrical frame with inclined legs. In Equation (10) the change of the length of the various members is considered zero. Instead of using the dimensions given in Fig. 4, those shown in Fig. 22 will be used so as to compare the results with a similar frame analyzed by a different method.[28]

Substituting in Equation (10) and reducing, and noting that, since the bases are fixed: $0 = 0.005602\ H_{1w} + 2.770\ H_{1w} - 150\,900 - 66\,560$; that is, $H_{1w} = 30\,390$ lb.

The moment at the intersection of the inclined member and the top horizontal strut $= 30\,390 \times 4.765 = 144.8$ ft-kips and the moment at the base $= 30\,390 \times 9.96 - 13\,400 \times 31.16 = -\,114.8$ ft-kips.

On comparing these values with the results given by the writer elsewhere[28] it can be seen that they agree very closely, but it seems to the writer that for the solution of Example *B* the general equations[29] are preferable to most engineers. Substituting the symbols shown in Fig. 22 in Equation (10) and reducing:

$$M_{AD} = -\,\frac{P\,D_a\,\phi_i\,L\,(3\,D_a + 4\,d)}{2\,\left\{D^2_a\,(1 + 6\,\phi_i) + 12\,D_a\,\phi_i\,d + 8\,\phi_i\,d^2\right\}} \quad \ldots\ldots\ldots(135)$$

[27] "The Design of Steel Mill Buildings", by the late M. S. Ketchum, Hon. M. Am. Soc. C. E., Fourth Edition, Rewritten, Equation (56), p. 303.

[28] *Proceedings*, Am. Soc. C. E., May, 1936, pp. 793–795.

[29] *Loc. cit.*, Equations (41) and (42), p. 795.

and,

$$M_{DA} = -\frac{P D_a L \left\{ D_a (1 + 3\phi_i) + 2\phi_i d \right\}}{2 \left\{ D_a^2 (1 + 6\phi_i) + 12 D_a \phi_i d + 8 \phi_i d^2 \right\}} \qquad \text{(136)}$$

in which $\phi_i = \dfrac{I_a L_i}{D_a I_c}$.

All calculations in Example *B* were carried to four significant figures. If they had been carried to ten significant figures the moments would have been 144 699 ft-lb and − 115 089 ft-lb, respectively. As the results differ by a very small amount the results to four significant figures are satisfactory and, therefore, a 20-in. slide-rule will give results that are reliable.

In Example *C* the truss shown in Fig. 23 is analyzed to show how the results by the author's analysis for a truss with an inclined upper chord

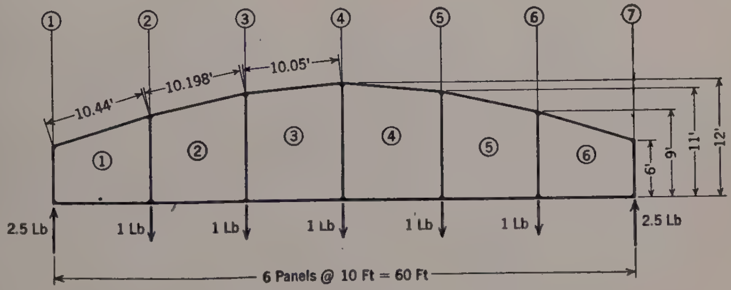

FIG. 23

compares with the results as calculated by relative deflections as proposed by Professor H. Yu.[30] The moments of inertia for all the members are equal.

The values of α and β are not known for an original design, but as the moments in the top and bottom chords have been determined by Professor Yu, these values have been used for determining the bending moment ratios, as shown in Table 2.

TABLE 2.—FACTORS FOR THE SOLUTION OF EXAMPLE *C*

Panel No.	Values of:				
	α	β	ϕ	$\frac{1+\phi\,\alpha}{1+\alpha}$	$\frac{1+\phi\,\beta}{1+\beta}$
1	0.9839	0.9862	1.044	1.022	1.022
2	1.003	0.9954	1.0198	1.010	1.010
3	1.011	1.0	1.005	1.003	1.003

Substituting the proper values from Fig. 23, and the proper factors from Table 2, in Equation (53), and reducing:

$$H_{1w} = H_{1w} \qquad \text{(137a)}$$

$$H_{2w} = 0.2963\, H_{1w} + 4.795\, H_{1w} - 0.6309\, M_2 + 2.944\, V_{12} \qquad \text{(137b)}$$

[30] "Stresses in Statically Indeterminate Structures", by Prof. H. Yu, National Wuhan Univ., Wuchang, Hupeh, China, Second Edition, 1935, pp. 486–494.

$$H_{3w} = 0.5477\, H_{2w} + 4.568\, (H_{1w} + H_{2w}) - 0.4553\, M_3 + 2.201\, V_{23} \quad (137c)$$

and,

$$H_{4w} = 0.7703\, H_{3w} + 4.609\, (H_{1w} + H_{2w} + H_{3w}) - 0.4005\, M_4 + 1.973\, V_{34} \quad (137d)$$

For the loading shown in Fig. 23: $M_2 = 25$ ft-lb; $M_3 = 40$ ft-lb; $M_4 = 45$ ft-lb; $V_{12} = 2.5$ lb; $V_{23} = 1.5$ lb; and, $V_{34} = 0.5$ lb. Substituting these values in Equations (137) and then expressing each value of H_w in terms of H_{1w}:

$$H_{1w} = H_{1w} \quad (138a)$$

$$H_{2w} = 5.0913\, H_{1w} - 8.4125 \quad (138b)$$

$$H_{3w} = 30.61356341\, H_{1w} - 57.94632625 \quad (138c)$$

and,

$$H_{4w} = 192.7543434\, H_{1w} - 367.5198853 \quad (138d)$$

From the symmetry of the loading in this particular example, it is obvious that $H_{4w} = 0$; hence, from Equation (138*d*), $H_{1w} = 1.90667$ lb and, from Equations (138*b*) and 138*c*), $H_{2w} = 1.29493$ lb and $H_{3w} = 0.423637$ lb.

After the H_w-values have been determined from Equation (53) the next step is to solve for the M_w-values for comparison. Applying Equations (67) and (69) to the first panel and reducing:

$$M_{2w} = 19.81 - 2.469\, M_{1w} \quad (139)$$

and,

$$V'_{12} = 0.4761\, M_{1w} - 1.171 \quad (140)$$

Applying Equations (66) and (68) to the second panel and reducing:

$$M_{3w} = -2.554\, M_{2w} - 0.9181\, M_{1w} + 9.181\, V'_{12} + 8.349 \quad (141)$$

and,

$$V'_{23} = 0.5674\, M_{2w} + 0.3\, M_{1w} - 3\, V'_{12} + 0.6575 \quad (142)$$

Applying Equation (66) to the third panel and reducing:

$$M_{4w} = -2.669\, M_{3w} - 0.8354\, M_{2w} - 0.8354\, M_{1w} + 8.354\, V'_{12} + 8.354\, V'_{23} + 5.907 \quad (143)$$

The value of V'_{34} is not required for evaluating M_{1w}, M_{2w}, and M_{3w}, and is, therefore, not given. Substituting Equations (139) and (140) in Equations (141) and (142):

$$M_{3w} = 9.759\, M_{1w} - 53.00 \quad (144a)$$

and,

$$V'_{23} = 15.41 - 2.529\, M_{1w} \quad (144b)$$

Substituting Equations (139), (140), and (144) in Equation (143):

$$M_{4w} = -41.969495\,M_{1w} + 237.953072 \quad \ldots\ldots\ldots\ldots (145)$$

From the symmetry of the loading in this particular example, it is obvious that $M_{4w} = 0$; hence, from Equation (145), $M_{1w} = 5.6697$ ft-lb. The value determined by Professor Yu is 5.6695 ft-lb. Substituting the value of M_{1w} in Equation (139) gives $M_{2w} = 5.8115$ ft-lb; Professor Yu[31] found it to be 5.8115 ft-lb. Substituting the value of M_{1w} in Equation (144*a*) $M_{3w} = 2.3306$ ft-lb; Professor Yu reported 2.3298 ft-lb.

From the foregoing values it is seen that the author's general equations give results that agree with those found by other methods; and they should agree, because no approximations are assumed except that in all the analysis in this discussion the change in the length of the members is considered equal to zero.

The value of M_{3w} as given is not the absolute maximum because, if there is no unit load at Panel Point 2, but unit loads at Panel Points 3, 4, 5, and 6, then the value of M_{3w} will be larger and the value given by Professor Yu is 3.3355 ft-lb.

For the symmetrical loading assumed, $M_{4w} = 0$; but if unit loads are placed at Panel Points 2 and 3, M_{4w} no longer $= 0$, and the value given by Professor Yu is -2.1459 ft-lb.

If it is assumed that the H_w-values act at the center of the vertical members, the moments are as follows: $M_{1w} = 1.90667 \times 3 = 5.7200$ ft-lb; $M_{2w} = 1.29493 \times 4.5 = 5.8272$ ft-lb; and $M_{3w} = 0.423637 \times 5.5 = 2.3300$ ft-lb. The corresponding values given by Professor Yu are: 5.7122 ft-lb; 5.8398 ft-lb; and, 2.3316 ft-lb, respectively. In this case, again, the two different methods agree very closely.

To solve for the maximum moments in all the members of a Vierendeel truss is a rather lengthy process no matter what method is used, but it appears that the author's method is as short as any.

L. C. MAUGH,[31] ASSOC. M. AM. SOC. C. E. (by letter).[31a]—In the United States the use of Vierendeel trusses has been opposed consistently (in print at least) for such reasons as lack of rigidity and economy or for the difficulties that are involved in the analysis, design, and construction of such monolithic structures. In actual practice, however, structures of this type have been built or, more frequently, designed as a more standardized type, such as in the bowstring and open spandrel arch, when the Vierendeel truss arrangement would sometimes be the simpler solution. This simplicity is especially noticeable when welded or reinforced concrete construction is used. The present trend in the adoption of various structural types indicates that there will be an increasing use of the quadrangular panel system by those engineers who appreciate both its advantages and disadvantages (because it certainly has both). For this reason a general discussion of some of the methods that are

[31] Asst. Prof. of Civ. Eng., Univ. of Michigan, Ann Arbor, Mich.

[31a] Received by the Secretary October 7, 1936.

applicable to the analysis of the quadrangular framework will be pertinent at this time.

In the analysis of the Vierendeel truss, as in any hyperstatic frame, the engineer has the choice of two different methods of approach: One involving the strain energy of the system or its counterpart, internal and external work, and the other, the various deformation methods that use the displacement of the joints. In this paper the author has used the first method to obtain a general form for the three algebraic expressions that will provide a system of minimum strain energy, these equations being expressed in terms of the resultant internal forces acting on the vertical members. Ordinarily, there will be three of these unknown forces in each of the strain equations but, by ingenious algebraic arrangement, certain groups of equations are expressed in terms of the horizontal components of the vertical members as the only forces, and two variables, α and β, that represent the ratio between the unknown end moments of the top and bottom chords. This algebraic arrangement, which is a characteristic feature of the method of Professor Vierendeel and Professor Young, is probably as convenient and simple as any results that can be obtained by the use of energy or work theorems. In fact, the author has done exceptionally well in the development and application of this method of approach. By assuming various values of α and β, the horizontal components of stress, H, in the vertical members can be computed with sufficient accuracy; but what is true for the H and V-components, the writer believes, is not necessarily true for the bending moments in the chord members.

Small errors made in the calculation of the H and V-values (errors that can easily be made because of the necessity of carrying the computations out to many decimals, or due to the rather indeterminate quantities, α and β) may produce relatively large errors in the end moments acting on the chord members Thus, for example, in Fig. 8, the author obtained values of $H_{1w} = 601.96$ lb and $V_{12} = 420$ lb, which, if applied at the mid-point of the first vertical, gives a moment at the right end of the chord member equal to: $M_{21} = (420)\ (10) - (601.96)\ (6) = 588$ ft-lb. If an error of -3% is assumed in H_{1w} and of $+3\%$ in V_{12}, the computed value of the chord moment will be: $M_{21} = (1.03)\ (420)\ (10) - (0.97)\ (601.96)\ (6) = 823$ ft-lb, or an increase of 40 per cent.

In view of the fact that, for structures with top and bottom chords of different rigidity, the values of α and β are affected by this variation of the chord moments, the writer believes that it would be difficult to obtain accurate numerical results. When the top and bottom chords in the same panel have the same $\frac{I}{l}$-value, then $\phi = \alpha = \beta = 1$, and a direct solution can be made regardless of the inclination of the chord members.

For Vierendeel trusses or for any rigid frame in which the members have considerable variation in rigidity and in which the joints undergo relatively large linear displacements as well as rotation, the writer has frequently combined the use of deformation equations with auxiliary force systems to obtain a direct method of solution. With this method of approach, the various forms of successive approximations, such as the method of iteration or moment

distribution, can be incorporated. Various forms of the deformation methods are in general use, but the advantage of introducing auxiliary force systems to control the motion of the structure does not seem to be so well known.

To illustrate the use of auxiliary forces, the truss shown in Fig. 8 will be subjected to various systems that will allow only one linear displacement at a time. Thus, in Fig. 24(*a*), there is one vertical displacement Δ_1, and in

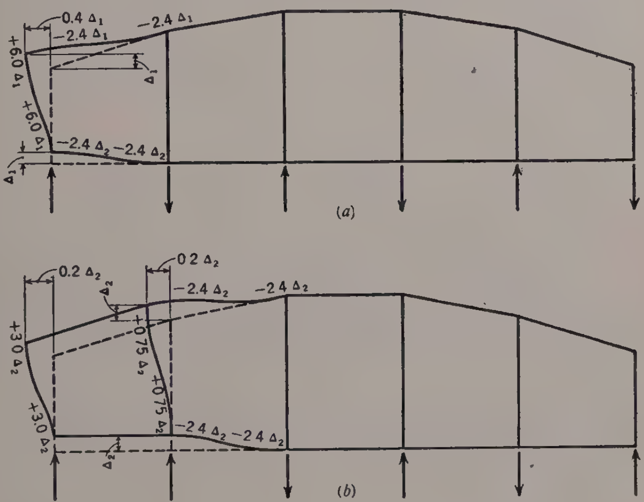

FIG. 24

Fig. 24(*b*), a vertical displacement, Δ_2. These displacements will produce end moments as shown when no rotation of the joints is considered and when a value of $I = 40$ is used. The final moments that will be developed for each value of Δ when the necessary rotation of the joints is allowed can easily be determined by moment distribution, or by calculating the angular rotation, using the method of iteration. In general, there will be as many of these special problems as there are panels in the truss although advantage can be taken of symmetry, as in the truss used by the author only three problems need be solved.

After the values of the moments that are consistent with each value of Δ have been obtained, an equilibrium equation for the moments in each panel for any desired load can be written easily. For the truss considered, with $I = 40$, the writer obtained the following set of equilibrium equations:

$$-15.62\,\Delta_1 - 1.53\,\Delta_2 - 0.276\,\Delta_3 + 0.044\,\Delta_4 - 0.004\,\Delta_5 + 6\,000 = 0 \quad \text{..(147}a\text{)}$$

$$-1.65\,\Delta_1 - 7.64\,\Delta_2 + 1.64\,\Delta_3 - 0.261\,\Delta_4 + 0.042\,\Delta_5 + 6\,000 = 0 \quad \text{..(147}b\text{)}$$

$$-0.272\,\Delta_1 + 1.6\,\Delta_2 - 5.96\,\Delta_3 + 1.6\,\Delta_4 - 0.272\,\Delta_5 - 4\,000 = 0 \quad \text{..(147}c\text{)}$$

$$+0.042\,\Delta_1 - 0.261\,\Delta_2 + 1.64\,\Delta_3 - 7.64\,\Delta_4 - 1.65\,\Delta_5 - 4\,000 = 0 \quad \text{..(147}d\text{)}$$

and,

$$-0.004\,\Delta_1 + 0.044\,\Delta_2 - 0.276\,\Delta_3 - 1.53\,\Delta_4 - 15.62\,\Delta_5 - 4\,000 = 0 \quad \text{..(147}e\text{)}$$

Equations (147) satisfy all the requirements for quick convergence when solved by the method of iteration. The following values were obtained after performing the numerical calculations for five cycles: $\Delta_1 = 337$; $\Delta_2 = 583$; $\Delta_3 = -696$; $\Delta_4 = -652$; and $\Delta_5 = -180$. These relative displacements are, of course, E times the actual values. For different applied loads the only change in Equations (147) is in the constant terms.

The foregoing method of solution requires considerable numerical work, but it has the following advantages: All calculations can be performed with sufficient accuracy by means of a slide-rule; the solution is direct and accurate for any arrangement of the truss; and corrections can be made quickly whenever errors are discovered.

For the Vierendeel truss in which the top and bottom chords in each panel have the same $\frac{I}{l}$-factor, as in the truss just considered, the writer prefers to make the analysis by considering each panel as a separate structural unit. This method has already been explained in other papers[32] and need not be repeated here. The procedure that has been outlined in this discussion is much more general than the panel method and can be applied to many types of problems. In fact, the wide scope of this method of analysis is not always appreciated.

[32] "The Analysis of Vierendeel Trusses by Successive Approximations", Publications of the International Assoc. of Bridge and Structural Engrs., Vol. 3, (1935).

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SIMULTANEOUS EQUATIONS IN MECHANICS SOLVED BY ITERATION

Discussion

By Messrs. Garrett B. Drummond, and A. W. Fischer

Garrett B. Drummond[4], Esq. (by letter)[4a]—A method of practical utility for the solution of simultaneous equations, is demonstrated in this paper. The steps for solving n simultaneous equations containing n unknown quantities, by the method of successive approximations, can be summarized as follows:

(1) Choose one equation in which the coefficient of one unknown is quite large in comparison with the coefficients of the other unknowns;

(2) Let all the unknowns, except that with the largest coefficient, assume the value zero;

(3) Solve for a first approximate value of this unknown;

(4) Choose another equation in which the coefficient of one of the unknowns, other than that chosen in Step (1) is quite large in comparison with the coefficients of the remaining unknowns;

(5) Substitute the first approximate value of the unknown already determined, allowing the remaining $n - 2$ unknowns to assume the value zero;

(6) Solve for a first approximate value of the second unknown;

(7) Continue in this manner, using first approximate values of each unknown, until first approximate values are determined for all unknowns; and,

(8) Repeat Steps (1) to (7) until the desired degree of approximation is reached, or until the difference between successive approximate values becomes negligible.

A very simple demonstration of this process can be given by solving the three-moment equations of the beam shown in Fig. 12. The equations are:

$$15\ M_2 + 54\ M_3 = -\ 132\,480 \qquad (54)$$

Note.—The paper by W. L. Schwalbe, Esq., was published in August, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the paper.

[4] With U. S. Engr. Dept., Memphis, Tenn.

[4a] Received by the Secretary September 4, 1936.

[4b] Corrections for *Transactions:* In the line below Equation (2), change "moment areas below the * * *" to "moments of the areas below the * * *"; in Fig. 2, omit the symbol "Q" from "Moment areas, Q"; and, in Equation (28) change the second quantity to read "$2\ (K_s + K_{s+1})\ \theta_s$".

and,

$$50\ M_2 + 15\ M_3 = -\ 104\,820 \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

From Equation (54): $54\ M_3 = -\ 132\,480$; and, $M_3 = -\ 2\,446$, which is the first approximate value of M_3. Substituting this value into Equa-

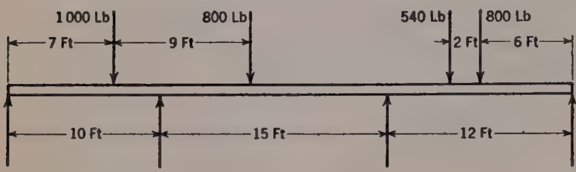

FIG. 12

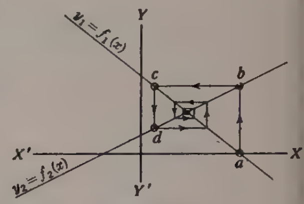

FIG. 13

tion (55): $50\ M_2 + (15)\ (-\ 2\,446) = -\ 104\,480$; or (simplifying and solving for M_2): $M_2 = -\ 1\,356$, which is the first approximate value of M_2.

Returning to Equation (54) with this value: $(15)\ (-\ 1\,356) + 54\ M_3 = -\ 132\,480$; and solving, $M_3 = -\ 2\,077$, which is the second approximate value of M_3.

Continuing in this manner, the second approximate value of M_2 is found to be $-\ 1\,473$; the third approximate value of M_3 is found to be $-\ 2\,041$; and the third approximate value of M_2 is found to be $-\ 1\,484$. These third approximate values are identical with the exact values determined from the ordinary simultaneous solutions of the two equations.

This apparently trivial problem was selected because it is possible to show, graphically, the precise nature of the steps in the approximate solution of two equations. Consider the two functions, $y_1 = f_1(x)$ and $y_2 = f_2(x)$, as shown in Fig. 13. The process was to begin at Point a, where $y = 0$ in Equation (54). With the corresponding value of x, the next step is to find the value of y in Equation (55)—in other words, proceed to Point b, Fig. 13, with this value of y; the third step is to find the value of x in Equation (54)—that is, move to Point c, Fig. 13. Continuing in this manner, with the successive values found, from the graphical standpoint, one merely follows the path shown in Fig. 13, which approaches closer and closer each time to the intersection of the graphs of the two equations, or to the numerical values of the simultaneous solutions of the equations.

Of course, from its practical use, the purpose of this method of solution is to lighten the labor of solving a large number of simultaneous equations, such as are encountered in indeterminate structures. This raises the question of the desirability of "exact" solutions. Just what is meant by "exact" solutions?

Setting up the equations, say, for the solution of a frame by the method of slope deflection, is it not true that certain assumptions are made regarding the action of forces at joints, and is the assumption not also made that the moment of inertia of the various members are constant throughout their cross-section? Then, it appears to be stretching the point to insist that the

equations resulting from these assumptions be solved for "exact" values. Approximate values are sufficient, and in arriving at such values the method explained by Professor Schwalbe is approximate and entirely applicable.

A. W. FISCHER,[5] ESQ. (by letter).[5a]—A rather unique method for the solution of simultaneous equations is presented in this paper. The simple examples are especially valuable, but the method is just as simple for solving equations of a more complicated nature. For example, it lends itself readily to the solution of the end moments that occur in the members of a triangular truss having rigid joints, caused by the deflection of the truss. After the moments at the ends of the members are computed it is an easy matter to calculate the secondary stresses caused at the end of each member.

Consider, for comparison, the Pratt truss shown in Fig. 14, which has been analyzed elsewhere[6] by Sophus Thompson, Assoc. M. Am. Soc. C. E., and

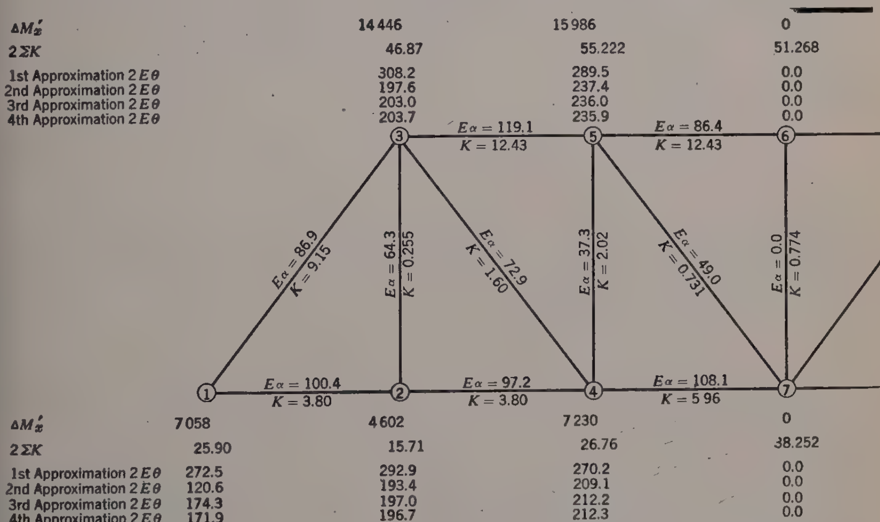

FIG. 14

Ralph W. Cutler, Jun. Am. Soc. C. E. Adding all the clockwise fixed-end moments and subtracting the counter-clockwise fixed-end moments at each joint and dividing them by $2 \Sigma K$ for all the members entering the joint the first approximation of $2 E \theta$ is determined for the various joints. For Joint 1, the sum of all the clockwise fixed-end moments[6] $= 4\,768 + 2\,290 = 7\,058$, and $2 \Sigma K = 2\,(9.15 + 3.80) = 25.90$. From this, the first approximation of $2 E \theta$ at Joint 1 $= \dfrac{7\,058}{25.90} = 272.5$. For Joint 6 the sum of all the fixed-end clockwise moments minus all the counter-clockwise fixed-end moments $= 6\,440 - 6\,440 = 0$, and $2 \Sigma K = 2\,(12.43 + 0.774 + 12.43) = 51.268$. From this value the first approximation of $2 E \theta$ at Joint 6 $= 0$. All the first

[5] Care, Pennsylvania Sugar Co., Philadelphia, Pa.

[5a] Received by the Secretary October 10, 1936.

[6] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), pp. 108–110.

approximate $2\,E\,\theta$-values are shown in Fig. 14. The second approximate value of $2\,E\,\theta$ for Joint 1 $= 272.5 - \frac{1}{25.90}\,(9.15 \times 308.2 + 3.80 \times 292.9) = 120.6$. The second approximate value of $2\,E\,\theta$ for Joint 2 $= 292.9 - \frac{1}{15.71} \times (3.80 \times 120.6 + 0.255 \times 308.2 + 3.8 \times 270.2) = 193.4$. Proceeding in this manner from joint to joint, in the order numbered in Fig. 14, all the second approximate values of $2\,E\,\theta$ are calculated and are shown in the diagram.

The third approximate value of $2\,E\,\theta$ at Joint 1 $= 272.5 - \frac{1}{25.90}\,(9.15 \times 197.6 + 3.80 \times 193.4) = 174.3$. Thus, all the third approximate values of $2\,E\,\theta$ are calculated (see Fig. 14). The fourth approximate values are then determined and as these values of $2\,E\,\theta$ at all the joints agree fairly closely with the third approximate values, there is no need to continue the operations. Values of $E\,\alpha$ can be computed by drawing a Williot diagram, then, applying the well known slope-deflection formula:

$$M_{AB} = 2\,E\,K_{AB}\,(2\,\theta_A + \theta_B - 3\,\alpha_{AB}) \quad \ldots\ldots\ldots\ldots\ldots (56)$$

the moments at the end of all the members can be solved. Equation (56) will give the moment at End A for any member, A-B. In this case, the sign convention is the same as that recommended by W. M. Wilson, F. E. Richart, and Camillo Weiss, Members, Am. Soc. C. E.[7] The moments at the ends of all the members can also be solved by the formula:

$$M_{AB} = 2\,E\,K_{AB}\,(2\,w_A + w_B) \quad \ldots\ldots\ldots\ldots\ldots\ldots (57)$$

TABLE 2.—COMPARISON OF MOMENTS BY THE AUTHOR'S METHOD AND BY THE EXACT METHOD

Member (see Fig. 14)	Length, L, in inches	Moment of inertia, I	Displacement, D, times modulus of elasticity	Moments: Exact method	Moments: Author's method (fourth approximation)	Member (see Fig. 14)	Length, L, in inches	Moment of inertia, I	Displacement, D, times modulus of elasticity	Moments: Exact method	Moments: Author's method (fourth approximation)
(1)	(2)	(3)	(4)	(5)	(6)	(1)	(2)	(3)	(4)	(5)	(6)
1-2.....	320	1 218	32 130	+230	−235	4-5....	372	750	13 870	−875	+882
2-1.....	....	....	....	+185	−141	5-4....	....	....	....	−924	+930
1-3.....	490.7	4 490	42 630	−230	+239	4-7....	320	1 907	34 600	+1 342	−1 335
3-1.....	....	....	....	−527	+530	7-4....	....	....	....	+2 602	−2 600
2-3.....	372	95	23 900	− 53	+ 54	5-7....	490.7	358	24 040	−130	+130
3-2.....	....	....	....	− 55	+ 56	7-5....	....	....	....	+42	− 42
2-4.....	320	1 218	31 100	− 80	+ 86	5-6....	320	3 978	27 650	+562	−579
4-2.....	....	....	....	−141	+145	6-5....	....	....	....	+3 544	−3 511
3-4.....	490.7	805	35 780	−307	+292	6-7....	372	288	0	0	0
4-3.....	....	....	....	−320	+305	7-6....	....	....	....	0	0
3-5.....	320	3 978	38 100	+900	−886						
5-3.....	....	....	....	+495	−486						

[7] "Analysis of Statically Indeterminate Structures by the Slope Deflection Method", *Bulletin No. 108*, Eng. Experiment Station, Univ. of Illinois, Urbana, Ill.

From the values given in Fig. 14, all the moments at the end of each member can be calculated and the results are given in Table 2. (In their Williot diagram[8], Messrs. Thompson and Cutler give a value of the displacement of 38 470 for Member 3–4. For a fixed-end moment of 700 as shown[9] the correct value would be 35 780, which is the value used by the writer.)

For purposes of comparison the values of the end moments by an exact method[10] is also given in Table 2 and on comparing these values with those computed by Professor Schwalbe it is seen that the moments by the author's method agree very closely. The signs shown are different for each member for the two methods, due to a different sign convention assumed. Furthermore, it should be made clear that, in Column (4), Table 2, the values of D times E are displacements (at right angles to the axis of the member) of one end relative to the opposite end.

From the foregoing it seems that the author's method can be used to solve for the end moments of all the members in a triangular truss with rigid joints in a very short and simple manner. The analysis is such that the results with a 20-in. slide-rule are reliable.

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[8] *Transactions*, Am. Soc. C. E., Vol. 96 (1932), Fig. 52 and Table 17.

[9] *Loc. cit.*, Fig. 53.

[10] "Modern Framed Structures" by Messrs. Johnson, Bryan, and Turneaure, p. 440.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

SIMPLIFIED METHOD OF DETERMINING TRUE BEARINGS OF A LINE

Discussion

By Messrs. Earl F. Church, Paul E. Wylie, James B. Goodwin, C. H. Swick, Philip Kissam, and George D. Whitmore

Earl F. Church,[3] Assoc. M. Am. Soc. C. E. (by letter).[3a]—Modern revolutionary developments in the science of surveying, in instruments used, in methods followed, and in results required, are making heavy demands on the technical training of surveyors. Practical astronomy is one phase of the subject in which proficiency would seem imperative. Certainly every surveyor should understand thoroughly the astronomic triangle (sometimes called the *PZS*-triangle) and its applications; the full significance of its six elements; which of the six can be found in an ephemeris; which of them can be observed; the spherical trigonometry methods of calculating any missing elements when three are known; and the practical use of the six elements in geodetic astronomy. In fact, these very points, together with the various time transformations, actually constitute the basis for a working knowledge of practical astronomy, which should be one of the working tools of every surveyor. Any work, such as the preparation of Mr. Inch's paper, in so far as it assists engineers in fulfilling the demands of modern surveying practice, is to be highly commended.

Free use of astronomic methods for azimuth determination, is certainly to be encouraged. Doubtless, it is fitting that the author should recommend the altitude method of making the observation, for the computations are simpler than those for the hour-angle method. If the altitude method of observing is used, there are three methods of computation to be considered: (1) The method described by Mr. Inch, utilizing the special table (expanded Table 1); (2) logarithmic computation of the desired azimuth, using a different form of the astronomic triangle formula better adapted to logarithmic calculations than that shown in the paper; and (3) computation of the desired

Note.—The paper by Philip L. Inch, Assoc. M. Am. Soc. C. E., was published in September, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the paper.

[3] Associate Prof. of Photogrammetry, Coll. of Applied Science, Univ. of Syracuse, Syracuse, N. Y.

[3a] Received by the Secretary September 24, 1936.

azimuth by means of the same formula shown in the paper, using a calculating machine, and without using Table 1. The writer proposes to discuss these three methods.

(1).—Computation of Azimuth as Recommended by the Author.—The surveyor would require a table of angular functions, a solar ephemeris, and the data presented by Table 1. The computation itself is shown in the paper and is not reproduced here.

(2).—Computation of Azimuth by Logarithmic Calculation of the Astronomic Triangle.—The surveyor would require a logarithmic table and a solar ephemeris, including a table of refraction and parallax corrections, but not the auxiliary table given in the paper. The formula used is:

$$\sin \frac{1}{2} Z = \sqrt{\frac{\sin \frac{1}{2} (z + \phi - \delta) \cos \frac{1}{2} (z + \phi + \delta)}{\cos \phi \sin z}} \qquad \text{.........(3)}$$

(in which, in addition to the notation of the paper, z = zenith distance), and the computation follows:

Observed h	=	25° 25′ 30″
Refraction	=	−0° 2′ 01″
Parallax	=	+0° 0′ 08″
h	=	25° 23′ 37″
z	=	64° 36′ 23″
ϕ	=	38° 53′ 40″
$z + \phi$	=	103° 30′ 03″
δ	=	−1° 02′ 16″
$z + \phi - \delta$	=	104° 32′ 19″
$z + \phi + \delta$	=	102° 27′ 47″
$\frac{1}{2} (z + \phi - \delta)$	=	52° 16′ 10″
$\frac{1}{2} (z + \phi + \delta)$	=	51° 13′ 54″

log sin $\frac{1}{2} (z + \phi - \delta)$	=	9.89812
log cos $\frac{1}{2} (z + \phi + \delta)$	=	9.79670
colog cos ϕ	=	0.10885
colog sin z	=	0.04413
	2)	9.84780
log sin $\frac{1}{2} Z$	=	9.92390
$\frac{1}{2} Z$	=	57° 03′ 45″
Z	=	114° 07′ 30″
		S65° 52′ 30″W
		64° 52′ 30″
		S 1° 00′ 00″W

(3).—Computation of Azimuth by Solving the Astronomic Triangle by Natural Functions and a Calculating Machine.—The surveyor would require a table of natural functions of angles, a solar ephemeris, and a calculating machine, but not the special table (expanded Table 1).

Equation (1) is used for this case and the entire computation is as follows:

Observed h	=	25° 25′ 30″
Refraction	=	−0° 2′ 01″
Parallax	=	+0° 0′ 08″
h	=	25° 23′ 37″
sin δ	=	−0.01811
cos h	=	0.90341
cos ϕ	=	0.77830
tan h	=	0.47470
tan ϕ	=	−0.80674

ϕ	=	38° 53′ 40″
δ	=	−1° 02′ 16″
$\dfrac{\sin \delta}{\cos h \cos \phi}$	=	−0.02576
tan h tan ϕ	=	0.38296
cos Z	=	−0.40872
Z	=	65° 52′ 32″
		64° 52′ 30″
		S 1° 00′ 02″W

It will be noted that by far the simplest calculation is the third, by means of natural functions with a calculating machine, but without the special table given in the paper. It will also be noted that the logarithmic calculation without Table 1 is really no longer or more difficult than the computation shown in the paper by means of the special auxiliary table.

The conclusions, therefore, are obvious: (1) The altitude method of determining azimuths by solar observations is simple, and its use is to be encouraged; (2) the field procedure described by the author is entirely adequate; (3) the quickest and most convenient method of computation, however, is by means of the formula, Equation (1), using natural functions and a calculating machine, but without the special table; and (4) further development of Table 1 for the purpose of facilitating the computation is scarcely worth while, because the calculating machine method, without an auxiliary table, is more accurate, and is easier.

PAUL E. WYLIE,[4] M. AM. SOC. C. E. (by letter).[4a]—A valuable short method of solving the astronomical triangle for azimuth is presented in this paper. It would be interesting if Mr. Inch would add data concerning the effect upon the accuracy of the result due to his assumption that A and B each vary in direct proportion throughout 1° of altitude.

Modern methods of navigation are based upon the solution of this same astronomical triangle, and much ingenuity has been expended upon labor-saving tabulations for the purpose. The navigator needs the azimuth of the sun (or other heavenly body) not only for the determination of compass error, but also in order to lay down upon the chart, in the proper direction, the line of position on which his ship is located. He does not, however, require azimuth within an error which is less than the error of plotting; consequently, most navigators' tables are not adapted to the use of the engineer.

To this rule there is at least one exception. A table[5] compiled by Lieut. Arthur A. Ageton, U. S. N., involves no knowledge more complicated than the addition or subtraction of two numbers. There is no interpolation and no multiplication. One addition and one subtraction of tabular values suffice to determine the azimuth, with a maximum error of less than a half minute of arc.

In the writer's opinion, Lieut. Ageton's method is as rapid in use as that of the author, and it probably presents even less opportunity for error. Moreover, it is already available at any agency of the Hydrographic Office at (to quote the statute) "the cost of printing and paper." An example, adapted from "Problem III", in Lieut. Ageton's 50-page book, follows: On December 17, 1934, at Latitude 20° 10′ N.; Longitude 163° 33′ W., the altitude of the sun (corrected) was observed to be 43° 36.0′ at G. C. T. 21 hr. 45 min. 26 sec. Find the sun's azimuth.

[4] Structural Engr.-Builder, Los Angeles, Calif.

[4a] Received by the Secretary, September 28, 1936.

[5] "Dead Reckoning Altitude and Azimuth Table", by Lt. Arthur A. Ageton, U. S. N., Hydrographic Office, *Publication No. 211*, U. S. Navy Dept., Washington, D. C.

From the almanac, determine the local hour angle and the declination of the sun in the usual manner. Opposite each half minute of arc in the tables, two columns are found, marked A and B. Using the tables as noted:

Local hour angle	=	(arc) 16° 13.4′ E.	A 55 376 (using nearest tabular value)
Declination	=	23° 21.9′ S.	B 3 716
			59 092
Altitude	=	43° 36.0′	B 14 076
Azimuth = 159° 13.5′, corresponding to			A 45 016

The simple underlying theory is given in the publication, and need not be repeated herein. The Ageton method involves an accurate knowledge of the time, and a working familiarity with the co-ordinates of the celestial sphere. In this respect, it is inferior to the author's method, and is disqualified for ordinary or occasional use, but its virtues make it well worth the attention of surveyors who must determine astronomical azimuths habitually.

It is to be hoped that it will be possible to print the author's complete table. It may well render more cumbersome methods obsolete.

James B. Goodwin,[6] M. Am. Soc. C. E. (by letter).[6a]—Suggestions have been solicited for making Table 1 serve its purpose more efficiently, and the only criticism that the writer would think applicable is that, perhaps, it might be more convenient if the sines of the angles of the sun's declination were embodied in an additional column, together with an accompanying column of differences for each minute. The column of altitudes (Column (1)) would then be combined to include the declinations. It would then be necessary to refer to only one source for altitude, latitude, and declination factors.

Equation (1) may be written in another form, namely:

$$\cos Z = \frac{1}{\cos h \cos \phi} (\sin \delta - \sin h \sin \phi) \qquad (4)$$

where $\frac{1}{\cos h \cos \phi}$ is as defined in the paper and $\sin h \sin \phi$ replaces $\tan h \tan \phi$. However, it is questionable whether this is any improvement on the author's procedure.

It would appear that, in the selection of the declination, standard time has been used for the 75th Meridian. In some possible conditions the use of standard time might lead to appreciable errors in establishing azimuths, especially at or near the limits of standard time belts. As an illustration, assume two traverse lines on either side of the boundary between two standard time belts or zones, run generally northerly and relatively close to the division line. One would have azimuths derived from a time 1 hr different from the other, whereas the same declination for the sun should practically apply, if close to the boundary.

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[6] Toronto, Ont., Canada.

[6a] Received by the Secretary September 28, 1936.

Since the sine of the sun's declination is multiplied by $\frac{1}{\cos h \cos \phi}$, the effect on the resulting azimuth is apparent. It would seem more nearly precise to consider the longitude of the place of observation in determining the time for the sun's declination referred to Greenwich.

In the illustration as given, there would be a difference in time corresponding to 2° of longitude, assuming Washington to be on the 77th Meridian. It would be of interest to know why, in determining azimuths, preference appears to be given to the method outlined as against observations on Polaris.

C. H. Swick,[7] Esq. (by letter).[7a]—Simplification of engineering methods and computations is always desirable, but this is especially true when astronomical measurements are involved. Ordinarily, an engineer has little need to start or check his surveys by the use of astronomical determinations and, therefore, is likely to lack familiarity with astronomical methods and computations. The method proposed by Mr. Inch is quite simple to apply and, therefore, will probably prove useful in certain types of surveys. The factors which he proposes to obtain from the table are rather easily computed for the individual observations, and it is questionable whether the double interpolations can be made from the table any more readily and quickly than the factors could be computed directly. The table has the advantage, however, that gross errors in the computations are probably less likely to occur when it is used.

The form of table proposed by Mr. Inch is compact and easily followed. As published, the interpolation interval is large, and this results in some inaccuracy especially in those parts of the table where the tabulated differences change rapidly, but the table would become rather unwieldy to compute and publish if the interval were decreased to 10 min or even to 30 min.

The writer would suggest a careful checking of all entries in Table 1 before the completed table is finally published. Several of the values are rather inaccurate in the last decimal place given. The following errors are also subject to correction in *Transactions*: (*a*) Change both θ and λ to ϕ wherever they occur, as in Equation (1) and Table 1; (*b*) in the second line following Equation (1), change the numerator, sin δ, to 1; (*c*) change Equation (2) to read, $\cos Z = A \sin \delta - B$; (*d*) in Line 18 under "Field Procedure", change "lower" to "upper"; and (*e*) in the tenth line preceding "Conclusion", change "Multiply Factor A by δ" to read "Multiply Factor A by sin δ."

Philip Kissam,[8] Assoc. M. Am. Soc. C. E. (by letter).[8a]—There are many advantages inherent in the determination of azimuth by sun observations. The method is chiefly applicable to route surveys or traverses of this type where neither established azimuth control is available nor high precision necessary. Under average conditions an azimuth determined by this method will be within 0° 2′ of the correct value, depending on the latitude and time of

[7] Chf., Section of Gravity and Astronomy, U. S. Coast and Geodetic Survey, Washington, D. C.

[7a] Received by the Secretary October 7, 1936.

[8] Associate Prof. of Civ. Eng., Princeton Univ., Princeton, N. J.

[8a] Received by the Secretary October 8, 1936.

year. The observations require 10 min, or less, time in the field and about 20 or 30 min for computation. They can be made during the regular program of observations and should be made once every clear day. Polaris observations usually are made more successfully at night than in the late afternoon when Polaris is visible during working hours and the preparation necessary for night observations, including the transportation of the field party to the site, usually requires an hour or more, is inconvenient, and often interferes with the progress of the next day's traverse work.

Since the sun method is so useful and can be made so frequently, it is important that the computations should be made with least difficulty. The value of the paper by Mr. Inch is the saving of time for computation. It will be interesting to compare the two usual methods of computation with that of the paper: Method (*a*) involves a formula for use with a computing machine, as follows:

$$\cos Z = \frac{\sin \delta - \sin \phi \sin h}{\cos \phi \cos h} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

and Method (*b*) involves a formula for use with a table of logarithms, as follows:

$$\cot^2 \frac{Z}{2} = \frac{\sin (s - \phi) \sin (s - h)}{\cos s - \cos [s - (90 - \delta)]} \quad \ldots\ldots\ldots\ldots (6)$$

in which, in addition to the notation of the paper.

$$s = \frac{\phi + h + (90 - \delta)}{2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

The steps required for computation are shown in Table 3. It will be noted that Method (*b*) requires several additions and subtractions of angles (9) which require time and may cause blunders. The Inch method requires fewer operations, and the references to the table are more convenient.

TABLE 3.—COMPARISON OF METHODS OF COMPUTATION

Steps required	Method (*a*)	Method (*b*)	Inch method
References to a table	6	5	4
Interpolations	6	5	6
Multiplications or divisions	3	0	1
Other minor steps	0	9	3

The author mentions the necessity of a solar ephemeris for computation of the sun's declination. It might be well to note that the best type of ephemeris to use is one that gives the declination at civil time (as in the Nautical Almanac). It is necessary to have an ephemeris for the proper year. If a permanent declination table is desired, the reader is referred to the report entitled "Azimuth Determination", by E. F. Coddington, M. Am. Soc. C. E.[9] Tables in this *Bulletin* extend through the year 2000, but the necessary computation is slightly more laborious.

[9] *Bulletin No. 79*, Eng. Experiment Station, Ohio State Univ., Columbus, Ohio, September, 1930.

The "Field Procedure" suggested by Mr. Inch is not the only practical one available. An excellent method is described under the title, "Topographic Instruction of the U. S. Geological Survey."[10] The chief feature of this method is in bisecting the sun's disk rather than in bringing the cross-hairs tangent to the disk. Ten pointings are recommended instead of two. The field time necessary for the ten pointings is about 6 min so that the average of the ten pointings can be used without danger of errors due to the curved path of the sun. Both the aforementioned *Bulletins* recommend using a white card behind the eye-piece on which the sun's image appears with the cross-hairs visible across the image. This method is rapid, as the observer does not have to place his eye at the eye-piece, and easy to use as no special equipment is necessary. The eye-piece must be focussed for this method, but this is balanced by the time required to attach a colored glass or prism.

(There is a difference of opinion as to the relative accuracy of pointing the instrument at the edge of the sun's disk rather than bisecting it. One of the faults of a sun observation is the difficulty of pointing a moving object. Although the sun's disk is large, the writer believes that bisection is more accurate. It is very difficult indeed to bring the cross-hairs to the edge of the disk without bringing them into the disk. Any effort to bring a cross-hair to the edge of a target is fraught with difficulties, whereas the eye can bisect with surprising accuracy as long as the entire image is visible without shifting the eyeball. An experimental determination of the comparative accuracies might well be made. It is also noted that special cross-hair arrangements are available from many instrument makers in which intersecting wires form a square slightly smaller than the sun's disk and for which claims are made of greater accuracy of pointing.)

In connection with the field procedure described in the paper, Step (3), namely, "level the instrument with extreme care", it might be well to recommend leveling with the bubble tube attached to the telescope if such a bubble is available. No amount of reversals will eliminate the errors caused by the vertical axis not being exactly vertical.

GEORGE D. WHITMORE,[11] ASSOC. M. AM. SOC. C. E. (by letter).[11a]—Unquestionably, the method presented by Mr. Inch for computing solar azimuth observations is a time-saver. Actual tests indicate that by his method the sun's azimuth can be found in about one-half the time required for solving the formula by use of trigonometric functions. It must be realized, however, and prospective users should be so warned in the explanatory text, that the azimuth found by this short-cut method may be incorrect by as much as 1′ in azimuth, solely because the interpolated values for Factors A and B may be incorrect. This is because the tables give the A and B factors only for each degree of altitude and each degree of latitude. For practical reasons, the intermediate values must be based on straight-line interpolation, whereas actual intermediate values would follow a curved line.

[10] *Bulletin 788-C*, U. S. Geological Survey.

[11] Chf. of Surveys, Eng. Service Div., T.V.A., Chattanooga, Tenn.

[11a] Received by the Secretary October 28, 1936.

Taking the author's example for illustration, the azimuth obtained by solving the spherical trigonometry formula gives a result about 30″ lower than the one he secures by using the A and B tables. Other experiments showed that, under certain conditions and combinations, the azimuth secured by using the tables of this form could be in error by as much as 60″.

In Example 1, Factor B is taken for 38° of latitude, the actual latitude being 38° 50′ 40″. The author finds the value for Factor B by interpolating between 38° and 39°, taking the tabulated difference of 22.1 for 38° of latitude, multiplying this value by 53′ 40″, and adding the result to Factor B for Latitude 38°. (A closer result would have been secured, incidentally, had he found B by interpolating "downward" from 39°, the actual latitude being only 6′ 40″ less than 39°, compared with 53′ 30″ plus from 38°.) The point, however, is that the difference in the fifth place of logarithms for Factor B for 1′ at 38° is 22.1, and for 1′ at 39° is 22.7. It is apparent that this difference of 0.6 will create rather sizable differences in the values for Factors A and B, when the latitude is about midway between the even degrees.

Readers should be warned also that Factor B is perhaps a more critical value than Factor A. They will both be subject to some error, but whatever the error might have been in Factor A, as taken from the tables, its effect is considerably reduced when A is multiplied by sin δ, which may be any quantity between 0.0 and 0.4. Factor B is used in the formula, however, just as it is taken from the tables, and is not reduced in any manner. The explanation should include a warning also that the computed result will vary in accuracy with the seasons, since the greater the declination and the larger its sine, the less the error in Factor A is reduced. It also appears that a south declination is likely to give a greater error than a north declination, since the two factors, A and B, are always added when the declination is south.

Inexperienced computers should be warned as to the declination tables to be used. Some of the solar ephemeris tables published by instrument manufacturers, supposedly to be used in connection with solar attachments, give the declination as including refraction corrections. The author's table gives the values for A and B already corrected for refraction; hence declination tables used in connection with this method should never include refraction corrections.

The tables might be more conveniently arranged by using one full page for each degree of latitude, and giving A and B values for altitudes from about 10 to 49°. Under this arrangement, the page might be divided into four columns, each column covering a range of 10° of altitude. This arrangement might also prove to be of advantage to a survey party wishing to compute solar azimuths in the field, as only one or two pages of the tables would be required for any one locality, thus eliminating the need for carrying a full set of tables for all latitudes. For the benefit of inexperienced computers, it might be well to explain that the "Differences for 1′" are in units of the fifth decimal place.

Except for the questions raised herein, it is believed that the five objects stated in the "Synopsis" will be accomplished, except possibly Item (2), which

is to "obviate textbook reference." The tables and the necessary explanations suggested by the author will in themselves constitute a textbook, and will have to be present always when solar azimuth observations are to be computed. It is suggested also that the principal uses of the method may be by computers who are engaged in calculating many solar observations as a routine operation, rather than by the engineer making an occasional solar observation and computation. Such an engineer may not be aware of the existence of these tables, or may not have them in his library, and the few extra minutes required to develop the regular formula is not so important where only an occasional observation is required. On the other hand, the saving in time for a computer calculating many observations, day after day, may be important. If this is to be the principal use, then it might be well to expand the table, showing the factors for, say, every 10′ or 20′ of altitude, and similarly for latitude. Such expansion of the tables would eliminate also the previous comments on inaccuracy in the interpolated values for the A and B factors.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

THE MODERN EXPRESS HIGHWAY

Discussion

By Messrs. Fred Lavis, Joseph Barnett, G. E. Hawthorn, John F. Fairchild, Leslie R. Schureman, and C. H. Purcell

Fred Lavis,[4] M. Am. Soc. C. E. (by letter).[4a]—There can be little question that the facts and opinions expressed by the author should receive careful attention from engineers and others responsible for future design and construction of main trunk-line highways. This paper is timely and important. However, it raises a question that is fundamental. Mr. Noble seems to assume that it is the duty of the State (and by that is meant any governmental agency) to provide ample facilities for the operators of motor vehicles driven at high speeds on highways. It is a serious question, however, how far one should, or could, go in the expenditure of public monies for this purpose.

There is no doubt that future trunk-line highways, or future reconstructions of such highways, should be designed with reasonable consideration for future requirements, and at least one of these requirements is the operation of motor vehicles, trucks, buses, and private cars at high speeds.

Since the days of Wellington at the end of the Nineteenth Century, wherever proper engineering methods have been used, railway design has envisaged and provided for future traffic. In 1930 the writer[5] pointed out the desirability of applying Wellington's formulas, modified to suit the conditions, to the design of highways. Mr. Noble has amplified this suggestion to include safety of operation as well as the economic operating factors cited by the writer.

The fact must be borne in mind, however, that, although the highway accident record is bad and involves a considerable economic loss, and although it is in many ways a real disgrace, the development of high-speed operation imposes on the State a much more serious duty than that of providing a safe roadway—namely, the duty of carefully selecting, testing, and supervising the operators of those vehicles which are driven at high speeds. There is a

Note.—The paper by Charles M. Noble, Assoc. M. Am. Soc. C. E., was published in September, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the paper.

[4] Cons. Engr., New York, N. Y.

[4a] Received by the Secretary September 22, 1936.

[5] "Highways as Elements of Transportation", by Fred Lavis, *Transactions*, Am. Soc. C. E., Vol. 95 (1931), p. 1020.

further duty in the periodic inspection and testing of the vehicles themselves. Simply to provide speedways and let any one and every one use them who meets certain very minimum requirements—people whose responsibility is not frequently checked and verified—is really criminal and far more a subject of criticism than any charge that can be directed against the structure of the highways even as they exist to-day. Perhaps, some day the Utopia may arrive, in which all vehicles and drivers not meeting certain requirements will be limited automatically to certain comparatively low speeds.

An interesting commentary on this phase of highway safety is the statement of the Commissioner of Motor Vehicles of the State of New York of the revocation or suspension of licenses for the two-week period ended September 19, 1936. For the State the total was 733 and among the causes listed were: Driving while intoxicated; leaving scene of accident; failure to pay registration fee; false statements on application; reckless driving; speeding; etc.

The fact that 733 licenses or registration certificates were suspended or revoked in only two weeks in one State indicates the extreme importance of the human element and it is, of course, almost certain that many other drivers than those who were discovered and penalized are not competent to drive at all and certainly not at high speeds. It will be noted also that no mention is made in these lists of the physical condition of the vehicles being driven.

There is also another phase to be considered in connection with the author's proposals for roads with large radius curvature, comparatively smooth profiles with light rates of gradient, and smooth rigid pavements. From the standpoint of safety, the effect on the driver of this monotonously even surface should be taken into consideration. There is some evidence that it may tend to dull the senses and produce a state, at least, of semi-somnolence which does not develop in driving over roads on which vigilance is obviously necessary. It may be admitted that the tendency of the future is likely to be toward the construction of highways of the very high type suggested by Mr. Noble but this argument, in turn, brings the subject back to the greater need of testing, licensing, and supervising the operators of the vehicles using the modern highway, and the vehicles themselves.

One statement is made in the paper in regard to pavements to which, perhaps, more particular attention may be drawn. Considerable has been said and written about the effect of heavy motor vehicles on highway costs, with the fairly general assumption that there is some more or less definite relation between the two and that the cost of highways is increased in some more or less direct ratio to the weight of the vehicles using it. More definite, perhaps, is the assumption that pavements (as also, to some extent, the alignment gradients, and width of roads) are controlled by the requirements of heavy motor trucks.

In December, 1934, the writer pointed out[6] some of the fallacies of this assumption and it is of interest, therefore, to note the author's statement that safe operation of modern private cars requires such widths and thicknesses of pavements that they will be amply sufficient for the heaviest vehicles.

[6] "Effect of Heavy Motor Vehicles on Highway Costs", National Research Council, 1934.

In his "Conclusion", Mr. Noble states that "the highway engineer is facing a tremendous responsibility. He must have courage and vision." This sense of responsibility, courage, and vision have not been lacking, and the engineers of the United States have been far in advance of those of Europe in designing and building various types of so-called super-highways. It must be borne in mind that much of the policy of the development of transportation in Europe is based on military needs and cannot be assumed to be caused entirely by anxiety to provide for the speeding of private cars, although this latter may develop as an incidental factor.

The construction of the Westchester County Park System, in New York, incorporating the most advanced thought in widths of pavements, elimination of grade crossings, etc., was begun in 1913. The New York State Highway from the Holland Tunnel to Elizabeth, N. J. (Route 25), was envisaged in 1923 and construction was begun in 1924. Boston, Mass., Philadelphia, Pa., Chicago, Ill., and other centers have contributed ideas and proceeded with construction.

The real question is not the courage and vision of the engineer, it is the reasonable adaptation of construction to future needs without unduly taxing certain classes of people, or all the people, for the benefit of the few.

The question as to how far one should go in making the expenditures necessary for the extension of these super-highways, is one of statesmanship just as much as engineering. One suggestion offered by Mr. Noble may be emphasized and that is the desirability of advance planning of such highways and the acquisition of the necessary land, the restriction of encroachments, etc., so that when the proper time comes the actual construction will not be unduly burdensome.

It should be stated that the author has done a service in emphasizing some of the factors that should be given consideration if, as, and when, such highways are required and the funds for their construction are made available.

JOSEPH BARNETT,[7] M. AM. SOC. C. E. (by letter).[7a]—In reading this paper one is given the impression that "Express Highway", "High-Speed Highway", and "Dual Highway" are considered to be synonymous. They are not so necessarily.

An express highway is one which may be traveled without undue interruption between terminals or other designated control points, much as express vehicles in any other form of transportation. It need not be high speed, necessarily, but the speed is higher, generally, than is common on adjacent local highways. The Westchester County and Long Island Parkways, in New York, properly may be classified as express highways. They may not be traveled at high speeds as that term is understood now, but may be traveled without undue interruption, in comfort, and with a sense of the beauty of Nature not possible on high-speed highways.

Dual highways are composed of two one-way highways. They need not necessarily be high speed or express. All highways carrying more than 10 000

[7] Senior Highway Design Engr., U. S. Bureau of Public Roads, Washington, D. C.

[7a] Received by the Secretary, September 30, 1936.

vehicles daily should be classed as dual highways. Increased safety and only a small additional cost over that of four-lane two-way highways are sufficient justification. Expensive 300-ft rights of way are desirable, but far from necessary. The largest part of the possible additional safety is attained as soon as opposing traffic is separated physically, even if the separation is only 3 or 4 ft of curb. All additional effort and expenditure, such as wide landscaped areas between highways resulting in wide shoulders to aid in avoiding collisions, reduction of head-light interference, etc., are worth while but they are subject to the law of diminishing returns in providing additional safety.

It should be possible to travel at high speeds on all modern express highways. However, the fact that motor vehicles are being constructed to travel safely at 100 miles per hr is not in itself sufficient justification for the adoption of this speed for design. It is only one of the factors which should be considered in weighing the problem. Of great importance are the topography, which is a factor in the determination of cost, and economic justification. One of the most important transcontinental highways in the United States would be included by many in any program for developing modern express highways, and yet hundreds of miles of this highway, where not under the influence of local urban traffic, carry about 300 vehicles daily. A two-lane, two-way highway designed for a speed commensurate with the particular topography is all that is justified.

Where traffic justifies the construction of four one-way highways, two lanes in each direction for trucks and two lanes for passenger vehicles, many advantages accrue to locating the truck or slow traffic highways outside rather than inside the fast traffic highways. "Fly-under" crossings are expensive, especially in flat country and the entrances to the slow traffic high-

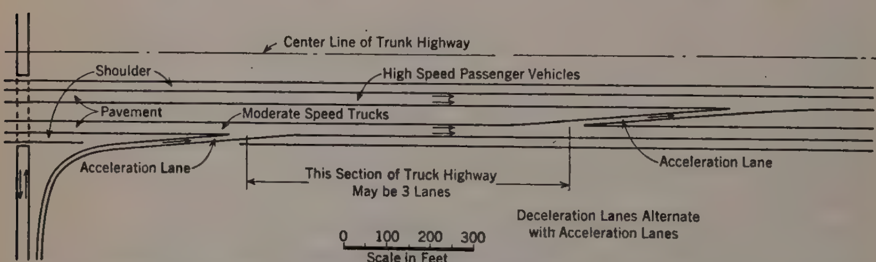

FIG. 5.—ACCESS PLAN FOR A FOUR-PAVEMENT TRUNK HIGHWAY. ONE QUADRANT IS SHOWN; OTHER QUADRANTS ARE SIMILAR.

ways are comparatively numerous. By placing the slow traffic highways on the outside, fast-moving passenger vehicles may use the slow-moving highways for short distances to enter and leave the dual highway. Fig. 2 would then be modified, as shown in Fig. 5.

Details of Design.—The complete formula for super-elevation (see Equation (1)) is,

$$E + F = 0.067 \frac{V^2}{r} \quad \text{(5)}$$

in which, in addition to the notation of the paper, F equals the side friction factor. On the assumption that there is an ample margin of safety against skidding when a vehicle rounds a curve at the minimum speed at which side pitch is noticed by driver or passenger, the U. S. Bureau of Public Roads has received reports on nearly 900 road tests to determine this speed. The results have not been analyzed thoroughly, but they lead to the conclusion that a side friction factor of 0.16 may be used with safety for vehicle speeds as great as 60 miles per hr. The results indicate lower factors for speeds greater than 60 miles per hr, but the combination of highway, vehicle, and driver capable of driving safely on curves at speeds greater than 60 miles per hr is rare, and few observations at higher speeds were received.

It is suggested that the minimum radius of curves be determined by the use of Equation (5); for example, for a speed of 60 miles per hr, a superelevation of $1\frac{1}{4}$ in. per ft, and a safe side friction factor of 0.16, the limiting radius is 915 ft. For a speed of 100 miles per hr and an assumed safe side friction factor of 0.08, the limiting radius is 3 640 ft. This method seems superior to one in which side friction is ignored, and a lower velocity is assumed to compensate for it.

The writer cannot urge too strongly the adoption of transition curves. Mr. Noble states that the necessity for such curves is doubtful because the vehicle is free to make its own transition, necessary within the wide lane. It is not sufficient that highways be designed so that it is possible for vehicle operators to remain in their lanes. They must be encouraged to do so if safety is to be built into the highways. "Cutting corners" is one of the important factors contributing to the number of accidents on two-way highways which constitute the largest part of the highway system of the United States. Vehicle operators "cut corners" regardless of whether or not it is possible for them to keep to the traveled lane, chiefly because of a natural tendency to reduce the shock of the sudden application of centrifugal force. They are further encouraged to "cut corners" by the fact that the change from a crowned to a superelevated cross-section is placed on the tangent because of the lack of a transition curve. To keep from sliding down the inclined cross-section, the operator must turn his wheel slightly against the direction of the curve ahead, a most unnatural action, the necessity for which encourages him to head toward the inside of the curve. With transitions, superelevation increases with curvature, and no complicated methods of changing from crowned to superelevated cross-sections are required. Reverse curves need no special treatment and the ends of adjacent transitions may be a common station. With adequate tables the design and location of curves with transitions are not complicated and require no more time and effort than curves without transitions.

The required length of transition depends upon the rate at which the final constant centrifugal acceleration is approached. If a vehicle travels at a constant speed on a curve it is accelerating toward the center at the rate of $\frac{V_s^2}{r}$, in which V_s represents the speed, in feet per second. The total time required

to traverse the transition curve is $\frac{L_s}{V_s}$, in which L_s represents the length of transition, in feet. The average rate at which the vehicle on the transition approaches this final constant centrifugal acceleration, therefore, is $\frac{V^2_s}{r} \div \frac{L_s}{V_s} = \frac{V^3_s}{r\,L_s}$. This average rate will vary for different drivers and numerous scientifically controlled tests are needed to determine the most desirable value. The few observations available indicate that a value of 2 for $\frac{V^3_s}{r\,L_s}$ is a fair average. Equating and changing to V, in miles per hour, results in the formula,

$$L_s = 1.58\,\frac{V^3}{r} \qquad (6)$$

For a speed of 100 miles per hr and a radius of 3 640 ft, the required length of transition is about 450 ft.

Regardless of the assumed design speed an alignment of curves with adequate transitions is greatly superior to an alignment of tangents and curves without transitions, in appearance and in ease and comfort of travel. Any one doubting this statement should operate a vehicle on a highway, such as the Mount Vernon Memorial Highway between Arlington and Mount Vernon, Va., on which all curves are constructed with transitions.

A deceleration of d = 17.4 ft per sec is equivalent to a uniform rolling friction factor between wheels and pavement of 0.54. Although this is not too great for a motor vehicle on clean pavements even when wet, it is felt that in computing minimum sight distances for one-way highways a friction factor that would apply to most vehicles on pavements not altogether clean should be used. This friction factor is likely to be about 0.4. The general equation[7b] for braking distance based on a uniform rolling friction factor between wheels and pavement follows:

$$s = \frac{V^2_s}{2\,g\,(F \pm S)} \qquad (7)$$

Converting Equation (7) by substituting V to represent the velocity, in miles per hour, and 32.2 for g, results in the following:

$$s = \frac{0.0334\,V^2}{F \pm S} \qquad (8)$$

For an assumed uniform friction factor of 0.4, a vehicle at 100 miles per hr on a level highway would stop at a distance of 835 ft after the brakes are applied. What Mr. Noble terms "lag" is the sum of two reactions times, brake reaction time, and what may be termed "awake" reaction time. Brake reaction time has been tested widely and varies between 0.5 sec and 1 sec for most vehicle operators. Awake reaction time may be defined as the time it takes

[7b] Correction to be made before paper is published in *Transactions*; In Equation (3), the numeral, 1, is to be omitted in the quantity, (1-S).

for a vehicle operator to come to the realization that the brakes must be applied. In this case, the operator must come to the realization that an object in his lane is stationary. It may take a few seconds to come to this realization if the object is a stationary automobile. Tests for awake reaction time are needed.

Conclusion.—The use of an assumed design speed of 100 miles per hr may be justified on many highways in many sections of the United States. Indeed, some engineers (notably those of the Oregon State Highway Department), are constructing highways designed for critical, as distinguished from assumed, design speeds as great as 100 miles per hr. The final determination to construct great systems of highways for this speed depends not on the vision and courage of the engineer, who is well endowed with these attributes, but on financial ability and justification.

However, to be safe, any highway on which the expected vehicle speeds are greater than 40 or 50 miles per hr should be designed with a refinement not thought necessary a decade ago. It is in this broad field that the principles outlined in Mr. Noble's excellent paper can be put to use. In many cases, little if any additional cost is involved. The principal requirements are more care in design and construction and the services of a high type of engineer, of which there is a plentiful supply.

G. E. HAWTHORN,[8] M. AM. SOC. C. E. (by letter).[8a]—The ever increasing speed of automobiles presents a serious problem for the highway engineer. There does not seem to be any definite upper limit for speed which automobile designers will build into their vehicles of the future. At the present (1936) rate of increase, automobiles capable of speeds of 150 miles per hr may be expected within the next 10 yr. Mr. Noble, as well as many other highway engineers, fixes the maximum speed for which express highways should be designed at 100 miles per hr.

A decade ago the highway engineer had the problem of ever-increasing weights of vehicles. This problem has been solved, at least temporarily, by rigidly enforced laws limiting maximum size and weights of vehicles. This has been accomplished by co-operation between automobile designers and users with highway engineers.

The problem of speed is similar but more serious, in that loss of life and destruction of property, instead of destruction of pavement only, are involved. Unless some rigid upper limit of speed is established, either by mutual agreement between motor-car designers and highway engineers or by a rigidly enforced law, engineers will continue to design highways that will become obsolete as to safety features every decade. There is some question as to whether it is desirable to design main highways for speeds as high as 100 miles per hr just because some automobiles can travel that fast. Possibly 70 miles per hr would be more desirable; certainly, it would be much safer. The driver's control over his vehicle varies, other conditions being equal, with the square of the speed. The vehicle traveling at 100 miles per hr requires twice the

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[8] Asst. Prof. in Civ. Eng., Univ. of Washington, Seattle, Wash.

[8a] Received by the Secretary October 1, 1936.

distance of one traveling 70 miles per hr, and four times the distance of one traveling 50 miles per hr to be brought under control in an emergency. All possible safety features that can be built into both the automobile and the highway can in no way change these ratios. They hold also as to the violence of the impact resulting from any accident.

Minimum speeds on express highways are also important. Even on dual highways fast traffic may be "boxed" by one slow vehicle passing another, resulting in serious accidents.

The two main safety features that can be built into the highway itself are: (*a*) Proper superelevation on all necessary curves; and, (*b*) sight distances of sufficient length to allow vehicles operating at maximum speeds to be brought under control or stopped at all times in case of an emergency.

Safety Feature (*a*).—The superelevation used on curves can be considered from several angles. Equation (1) of the paper gives the theoretical superelevation desirable for vehicle operation on curves with the same safety as on tangents. Unfortunately, the maximum superelevation that can be used safely, if the road is to be used by slow traffic, is limited approximately to a 10% cross-grade, which allows a maximum superelevation of about 0.1 ft per ft of width. Using this value, the minimum radius of curve that could be used would be:

$$r = 0.67\ V^2 \qquad (9)$$

or for speeds at 100 miles per hr, $r_{min.} = 6\,700$ ft. Automobiles can be operated on curves which are under-superelevated, however. Railroad engineers have determined that an under-superelevation of 3 in. in a gage width of 4.71 ft, may be used without discomfort to train passengers. Using this value for comfortable riding,

$$E = \frac{0.067\ V^2}{r} - 0.053 \qquad (10)$$

On curves where the superelevation is not enough, theoretically, the vehicle has a tendency to turn out and follow a curve of greater radius. This is an added hazard, particularly at high speeds. It can be shown that, if the deficiency in superelevation is equal to the coefficient of friction between tires and surface, vehicles will skid sidewise. At the skidding superelevation:

$$E = \frac{0.067\ V^2}{r} - f \qquad (11)$$

in which f is the coefficient of friction. Equation (11) is important only at high speeds or on slippery pavements.

Deficient superelevation may also cause overturning. It can be shown by the equilibrium of forces that the vehicle will overturn when,

$$E = \frac{0.067\ V^2}{r} - \frac{b}{2\ h} \qquad (12)$$

in which b = the width of the car from center to center of tires and h = the height of the center of gravity of the vehicle above the roadway.

Safety Factor (*b*).—The sight distance required for safety on highways depends on the distance the vehicle will travel while being brought under control or stopped. A fundamental formula for the required sight distance can be derived, using the relation of work and energy. Considering the kinetic energy of the rotating parts of the vehicle, its total kinetic energy is equal to $\frac{1.05\,W\,V^2}{2\,g}$, in which W = the weight. Then changing velocity from feet per second to miles per hour, $0.03511\ W\ V^2 = W\,f\,s + W\,S\,s$, and the required distance equals,

$$s = \frac{0.03511\ V^2}{f + S} \qquad (13)$$

in which S = the grade, in feet per foot (− = down grade and + = up grade). A term should be added to Equation (13) representing the distance traveled during the time used by the operator in applying the brakes. If s_a equals this time, in seconds,

$$s = \frac{0.03511\ V^2}{f + S} + 1.47\ s_a\ V \qquad (14)$$

For values of $f = 0.4$; $S = -0.05$ ft per ft; $V = 100$ miles per hr; and $s_a = 1$ sec. $s = 1\,002 + 147 = 1\,149$ ft. The rate at which the vehicle decelerates varies with the value of the coefficient of friction, f, and $d = 32.2\ f$, in which d = the deceleration, in feet per second per second. Substituting d for f in Equation (14):

$$s = \frac{0.03511\ V^2}{\dfrac{d}{32.2} + S} + 1.47\ s_a\ V \qquad (15)$$

The first term of Equation (15) can be readily reduced to Equation (4) of the paper.

Sight distance and superelevation are built into the pavement and can not be changed once it is laid. The only unknown factor in these equations is the speed. Automobile designers and users, together with highway engineers, should agree on a reasonable maximum speed for which the main highways and automobiles are to be designed. Then, and only then, can highways be designed which are as safe as it is possible to build them. Speed will still claim its victims, but it will not be the fault of the highway.

Wide smooth shoulders free from poles and signs, flexible guard-rails placed at reasonable distances from the edge of the surface, and the limiting of grades are desirable, but they become really important only when the automobile is out of control. They can be replaced or improved as traffic demands change, at relatively small cost. Great improvement has been made along this line in existing highways in the United States during the last few years.

JOHN F. FAIRCHILD,[9] ASSOC M. AM. SOC. C. E. (by letter).[9a]—A phase of this subject was omitted from the paper which, although it does not affect the construction of the highway itself, might affect its cost materially. For example, Fig. 1 shows a 300-ft right of way on the side lines of which appear the words "No Frontage or Access Permitted." In the paragraph following Fig. 1, Mr. Noble emphasizes this proposal by the statement that "* * * a safety factor is to prevent frontage of any type on the highway and to exclude farm and local road entry. It appears reasonable to restrict access to approximately 10-mile intervals."

If the express highway is in the location of an existing road, the construction of the highway under such restrictions would confiscate a very valuable right to the abutting properties for which compensation must be paid. If the express highway is located on an entirely new right of way, no doubt, it would cut some properties into separate parcels, so that it would be necessary to furnish under-crossings or over-crossings between the two parcels thus separated. This situation, of course, would vary in different localities, which fact should be taken into consideration in estimating the cost of the express highway.

LESLIE R. SCHUREMAN,[10] ASSOC. M. AM. SOC. C. E. (by letter).[10a]—The author renders a valuable service to the Highway Engineering Profession and to the general public in stimulating thought and discussion on what is probably the most pressing problem which confronts the highway engineer to-day —that of safety in highway design. However, any analysis of this problem cannot be complete without some consideration of the highway bridge which, although in itself a problem in design, must be included in any broad approach to the problem of determining and establishing standards of design for safe trunk highways. The accident record offers indisputable testimony in support of the fact that the highway bridge, as it exists to-day, is as hopelessly obsolete as the highway itself for the high-volume, high-speed traffic which must be provided for in the very near future.

The highway bridge engineer must develop design standards and details for a structure which will accommodate such traffic safely and efficiently. Its realization necessarily depends upon the co-operation of the highway engineer in providing straight-line tangents and continuity of grade and alignment at the site. Serious thought must be given to the roadway width of the structure. Obstruction to travel on the shoulder area constitutes a definite traffic hazard which can, and should, be eliminated. Pavement widening or flaring on approaches has proved to be a valuable safety feature. Approach slabs designed to span the back-fill area eliminate the startling end-of-deck bumps so common on present-day bridges. Balustrades must be structurally adequate to resist complete failure from collision and, preferably, should be provided with a first line of defense in the form of a curb. Poorly designed balustrades are much too frequently a contributing factor in serious accidents. Center piers on under-passes are a highly fertile source of motor-car disaster and,

[9] Harrington Park, N. J.

[9a] Received by the Secretary October 7, 1936.

[10] Asst. Prof., Civ. Eng., Princeton Univ., Princeton, N. J.

[10a] Received by the Secretary October 19, 1936.

unquestionably, should not be used except in locations where the underpassing highway is divided by a center strip of ample width. Low points in the highway profile frequently occur at stream crossings. Adequate deck drainage eliminates the hazard of a wet or icy pavement condition.

Quite aside from purely utilitarian considerations, the structure should undoubtedly possess true grace and beauty. The question of æsthetics has become increasingly important with the growing need for grade-crossing eliminations. The intelligent selection of materials and design of proportions rather than detailed embellishment and ornamentation contribute much to an æsthetically pleasing result. The æsthetic treatment of any structure constitutes a problem in itself and should certainly be given considerably more time and thought than, in most cases it has received in the past.

C. H. PURCELL,[11] ASSOC. M. AM. SOC. C. E. (by letter).[11a]—The line of thought developed throughout this paper is well worth serious consideration. The next stage in highway design will be along the lines of greater safety, although perhaps not to the extent that speeds of 100 to 115 miles per hr will be common. It is quite feasible to eliminate a considerable percentage of accidents now classed, and rightly so, as "fault of driver." This can be done simply by the expenditure of sufficient sums of money. The engineering knowledge and experience are available now and can be applied whenever the work can be financed. The width of lane, radius of curvature, rate of superelevation, etc., for any maximum speed can be determined at a cost that will be insignificant in comparison to the total cost of improvement required. It is very doubtful indeed if all "fault-of-driver" accidents can be eliminated, and certainly not by engineering effort.

Records indicate that approximately one-half of all motor vehicle accidents in California occur in cities. A fair percentage occur, also, on secondary roads. Construction of super-trunk line roads, therefore, can eliminate much less than one-half the problem. In fact, such construction might easily increase the accident rate in cities and rural areas through emphasis of the high-speed complex.

On the basis of records for the first six months of 1936, accidents involving two or more vehicles fall into three groups of vehicle types:

Type of accident	Percentage of total
Passenger *vs.* passenger	69.70
Passenger *vs.* freight	21.67
Freight *vs.* freight	2.67
All others	5.96

Grouped in reference to the course being pursued, these accidents appear in the order of frequency, as follows:

Type of accident	Percentage of total
Approaching on the same road	41
Overtaking on the same road	31
Paths intersecting, but on the same road	16
Paths intersecting, but traveling different roads	12

[11] Chf. Engr., San Francisco-Oakland Bay Bridge; State Highway Engr., Sacramento, Calif.

[11a] Received by the Secretary October 24, 1936.

From the foregoing, it appears that the separation of freight and passenger traffic would eliminate nearly 22% of accidents involving vehicles. The construction of divided lanes and the separation of grades of intersecting roads would reduce accidents by 53 per cent. Construction of pedestrian lanes and cross-overs would greatly reduce such accidents. To the extent that improvements of this kind were successful, the "fault-of-driver" type of accidents would be reduced. The extent of increase of accidents of the "overtaking", "side-swipe", etc., types which might result from increase in speed thus made possible, is entirely a matter of conjecture.

Accidents in which single vehicles are involved are grouped, as follows:

Type of accident	Percentage of total
Collision	35
Non-collision	46
Pedestrian	19

The statement, "drove off the road", accounts for nearly 25% of all single car accidents. What this percentage might be for vehicles traveling at speeds of 150 ft per sec, controlled by a driver with a reaction time of 0.5 to 1.5 sec, is problematical. Certainly, accidents which might result would be spectacular, and perhaps more likely to involve vehicles in a lane 100 or even 200 ft to the left than the farm house or other fixed object an equal distance from the right of way.

In connection with consideration of design speeds, it is interesting to note the detail secured during a survey of actual road speeds at about sixteeen locations, conducted by the Maintenance Department of the California Division of Highways. The speed of approximately 22 000 vehicles was recorded during daylight and of 11 000 at night. Fourteen of the locations were such that any desired speed could be maintained. The average speeds were, as follows:

Vehicle	Average Speeds, in Miles per Hour	
	In daylight	At night
Passenger	47	45
Trucks only	34	34
Buses only	50	50

Of the daylight traffic, 94.09% traveled at a rate of 55 miles per hr, or less, and, during darkness, 94.49% traveled at, or less than, that rate.

Before entering on a program looking toward ultimate development of highways adequate for speed of 100 miles per hr, there are other factors to be considered.

Are public officials prepared to admit, for example, that they are unable to control the careless, the reckless, and the incompetent drivers at present legal speeds? Are the present methods of control of traffic by "stop and go" and other mandatory signals and reasonable regulations a total failure in controlling traffic? Are they certain enough as to what the effect would be if construction were advanced to permit of higher speeds? Are they at all certain that the majority of drivers desire to travel at higher speeds than present

roads permit? Have the people reached a stage where they are prepared to submit to the degree of regulation which German authorities, for example, can enforce in the operation of their express roads? What will be the effect of further improvements in airplane operation and construction of machines which can compete, in convenience and first cost, with the automobile?

There have been a number of highway projects constructed along the lines which Mr. Noble has in mind, and others will be developed which, from an experimental point of view, will provide information on which to base design standards.

The problem at present appears to be more politic and sociological than engineering. Every highway engineer should be alive to the situation, not only from a professional point of view but from that of the citizen as well, in order to serve the need in the best possible manner.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

COMPARISON OF SLUICE-GATE DISCHARGE IN MODEL AND PROTOTYPE

Discussion

By G. H. Hickox, Assoc. M. Am. Soc. C. E.

G. H. Hickox[8], Assoc. M. Am. Soc. C. E. (by letter).[8a]—Those engineers engaged in the design of hydraulic structures by means of models have long felt the need of such comparisons as Mr. Blaisdell has presented. It has been assumed for too long a time that the actions of model and prototype were necessarily identical. In the majority of cases where large models have been properly constructed and where friction forces are negligible the correspondence between model and prototype is probably good, but, in general, quantitative confirmation has been lacking.

The author's test results show a gratifying correspondence with the observations on the prototype. Such differences as exist are not readily accounted for, but may be due partly to the fact that the model does not represent the prototype accurately. The model was placed in a straight flume and the water approached it directly, whereas in the prototype, the channel supplying water to the gate approached at right angles and immediately above it. The channel in the prototype is also somewhat wider above the gates than below them, whereas in the model, both parts of the channel were represented by the same straight flume. Part of the differences found may be due to the consequent difference in the effect of the velocity of approach. It would seem difficult to reproduce the up-stream head with certainty in the straight channel used. The measurement of tail-water elevation may also be open to some question. Although it is by no means certain that they are the only causes of discrepancy, it seems very probable that at least they are contributing factors.

In models in which both gravity and friction forces are effective, the friction requirement may be satisfied by considering both Froude's law and

Note.—The paper by Fred William Blaisdell, Jun. Am. Soc. C. E., was published in January, 1936, *Proceedings*. Discussion on this paper has appeared in *Proceedings* as follows: May, 1936, by Messrs. Raymond Boucher, and H. E. Hurst.

[8] Hydr. Engr., TVA Hydr. Laboratory, Norris, Tenn.

[8a] Received by the Secretary September 16, 1936.

Manning's equation. For similarity with respect to the gravity forces, by Froude's law,

$$v = \sqrt{l g} \quad \ldots\ldots (4)$$

in which v, l, and g (in accord with the author's notation) are the scale ratios of velocities, lengths, and accelerations due to gravity, respectively. Since g is commonly unity, Equation (4) may be written,

$$v = \sqrt{l} \quad \ldots\ldots (5)$$

For similarity with respect to friction forces, Manning's equation may be utilized thus:

$$v = \frac{l^{\frac{2}{3}}\ s^{\frac{1}{2}}}{n} \quad \ldots\ldots (6)$$

in which s and n are the scale ratios of slopes and roughnesses, respectively. Where geometrical similarity exists between model and prototype, s = unity, and Equation (6) becomes:

$$v = \frac{l^{\frac{2}{3}}}{n} \quad \ldots\ldots (7)$$

In order to satisfy the two conditions simultaneously, Equations (5) and (7) must give the same value for v; that is,

$$l^{\frac{1}{2}} = \frac{l^{\frac{2}{3}}}{n} \quad \ldots\ldots (8)$$

from which $n = l^{\frac{1}{6}}$. In the case of the Tremont gates, the sluices were made of ashlar masonry with a probable value of n equal to 0.017. For correspondence, therefore, the n of the model should be: $\frac{0.017}{15^{\frac{1}{6}}} = \frac{0.017}{1.57} = 0.0108$. This value of n can be obtained with smooth cement surfaces so that, in this case, it seems probable that the requirement for similarity was nearly satisfied.

AMERICAN SOCIETY OF CIVIL ENGINEERS
Founded November 5, 1852
DISCUSSIONS

BACK-WATER AND DROP-DOWN CURVES FOR UNIFORM CHANNELS

Discussion

By J. C. Stevens, M. Am. Soc. C. E.

J. C. Stevens,[5] M. Am. Soc. C. E. (by letter).[5a]—The computation of any form of water-surface profile—on sustaining or adverse slopes, under super-critical or sub-critical flow conditions, in either rough or smooth uniform channels—may be made quickly by the simple step-by-step procedure fully described and illustrated by the writer in his discussion[6] of the paper by Arthur E. Matzke, Jun. Am. Soc. C. E.[7]

The functions set up by the author are only applicable to channels of uniform cross-section (that is, rectangle, trapezoid, triangle, etc.), whereas by far the greater number of surface profile determinations must be made for channels of irregular cross-section. For the latter case computations must be made by a trial method one of which was suggested by the writer[8] in 1925. For uniform sections a simple direct computation will yield results amply accurate for all practical purposes without going through the maze of mathematical processes so laboriously set forth by the author. In fact, the simple step-by-step formula used by the writer is fundamentally more accurate than that offered by Professor Mononobe, because there are no basic assumptions and approximations entering into its composition. The only limitation is that sections shall be taken sufficiently near each other so that the surface profile may be considered a straight line between them.

Assumptions that wetted perimeters and areas are exponential monomials, and that the Chezy coefficient is constant, are not at all necessary; neither is it necessary to find neutral and critical depths and velocities. One merely fortifies himself with a table of Kutter coefficients (or Manning's, or any other

Note.—The paper by Nagaho Mononobe, M. Am. Soc. C. E., was published in May, 1936, *Proceedings*. This discussion is printed in *Proceedings* in order that the views expressed may be brought before all members for further discussion of the paper.

[5] Cons. Hydr. Engr. (Stevens & Koon), Portland, Ore.

[5a] Received by the Secretary September 22, 1936.

[6] *Proceedings*, Am. Soc. C. E., August, 1936, p. 950.

[7] "Varied Flow in Open Channels of Adverse Slope", by Arthur E. Matzke, *Proceedings*, Am. Soc. C. E., February, 1936, p. 193.

[8] *Engineering-News Record*, October 1, 1925, p. 550.

set of coefficients that he prefers) and a slide-rule and performs the simple operations shown in Table 9. This table shows the calculated profile of the back-water curve under discussion for the smooth rectangular channel. The

TABLE 9.—SURFACE PROFILE IN SMOOTH RECTANGULAR CHANNEL (Width, $B = 0.656$ ft; discharge, $Q = 0.173$ cu ft per sec; Kutter's $n = 0.009$; and $S_o = 0.00200$)

Depth of flow, D, in feet (1)	Hydraulic area, A, in square feet (2)	Average velocity, V, across a section (3)	Velocity head, $\frac{V^2}{2g}$ (4)	Specific energy, ϵ (5)	Increment of specific energy, $\Delta \epsilon$, between adjacent depths (6)	Wetted perimeter, P, in feet (7)	Hydraulic radius, R, in feet (8)
0.328	0.215	0.805	0.0100	0.3380	....	1.312	0.164
0.312	0.204	0.847	0.0112	0.3232	0.0148	1.280	0.160
0.296	0.194	0.892	0.0124	0.3084	0.0145	1.248	0.156
0.279	0.183	0.945	0.0139	0.2929	0.0155	1.214	0.151
0.262	0.172	1.005	0.0158	0.2778	0.0151	1.180	0.146
0.246	0.161	1.074	0.0180	0.2640	0.0136	1.148	0.140
0.230	0.151	1.145	0.0204	0.2504	0.0136	1.116	0.135
0.213	0.140	1.235	0.0237	0.2367	0.0137	1.082	0.129
0.197	0.129	1.340	0.0279	0.2249	0.0118	1.040	0.124
0.181	0.119	1.453	0.0328	0.2138	0.0111	1.016	0.116

TABLE 9.—(*Continued*)

Depth of flow, D, in feet (1)	Chezy's coefficient, C (9)	Friction slope, S'_f, at a given depth point (10)	Average friction slope, S_f between two depth points (11)	Difference, $S_o - S_f$, between bed slope and friction slope (12)	Distance, Δl, between adjacent depth points (13)	Distances, l, Along the Bed of a Channel, Measured from the Weir: Computed (14)	By laboratory test (15)
0.328	115	0.00030	....	....	....	......	......
0.312	115	0.00034	0.00032	0.00168	8.8	8.8	9.2
0.296	115	0.00039	0.00037	0.00163	8.9	17.7	18.2
0.279	115	0.00045	0.00042	0.00158	9.2	26.9	28.1
0.262	115	0.00052	0.00049	0.00151	10.0	36.9	37.6
0.246	114	0.00063	0.00058	0.00142	9.6	46.5	47.6
0.230	114	0.00075	0.00069	0.00131	10.4	56.9	58.5
0.213	113	0.00091	0.00083	0.00113	12.4	69.3	69.6
0.197	113	0.00114	0.00102	0.00098	12.0	81.3	83.3
0.181	113	0.00145	0.00130	0.00060	18.5	99.8	98.0

discharge is not given, but presumably may be calculated from $A_o V_o = 0.173$ cu ft per sec. From the neutral slope, radius, and velocity given, one may find that Kutter's $n = 0.009$. Using the formula,[6]

$$\Delta l = \frac{\Delta \epsilon}{S_f - S_o} \quad \text{.......................} \quad (35)$$

in which Δl = the distance between adjacent depths; S_f = friction slope; S_o = bed slope; and, $\Delta \epsilon$ = increment of specific energy between adjacent depths, or:

$$\Delta \epsilon = \left(D_1 + \frac{V_1^2}{2g}\right) - \left(D_2 + \frac{V_2^2}{2g}\right) \quad \text{..............} \quad (36)$$

The depths given, in inches, in Table 7 were converted to feet and used in Table 9. The slope, S'_f, is the friction slope at the depth point, whereas

S_f is the average friction slope between depths. For such problems very little difference is found whether one averages slopes, areas, hydraulic radii, or velocities between depth points.

In the application of Equation (35) the specific energy, ϵ, has increasing values down stream, because the bed slope is greater than the friction slope. Therefore, values of $\Delta\epsilon$ are negative as are also values of $(S_f - S_o)$, making Δl positive up stream. Columns (14) and (15), Table 9, show how closely the calculated distances above the dam agree with the distances observed in the laboratory.

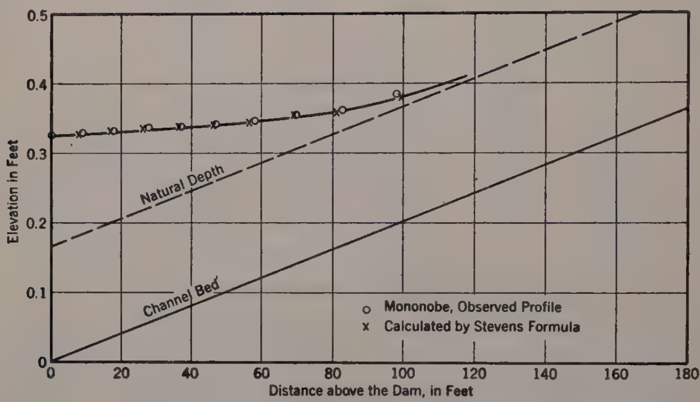

FIG. 20.—COMPARISON OF PROFILES FOR A SMOOTH RECTANGULAR CHANNEL

Fig. 20 shows the profile plotted from "Laboratory tests", Table 7, and the profile calculated by Equation (35). The agreement is practically perfect as such hydraulic calculations go.

A correction should be made to Figs. 14 to 19, inclusive, of the paper. The ordinates are not "values of D, in inches", but elevations, in inches, above the stream bed at the dam. D is the depth of water in the channel.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

FLOOD PROTECTION DATA
PROGRESS REPORT OF THE COMMITTEE

Discussion

By Messrs. Gordon W. Williams, Merrill Bernard, and Glenn W. Holmes

Gordon R. Williams,[14] Jun. Am. Soc. C. E. (by letter).[14a]—The floods in the years 1935 and 1936, which in some localities broke all previous records, make it very evident that the work suggested by the Committee in Recommendation (1) should be continued, both in the office and in the field. The accumulation of field data, especially the securing of flood discharge measurements, makes it possible to evaluate more closely, in terms of discharge, the data obtained when Recommendations (2) and (3) are followed. The flood discharge measurements made in 1936 will make it possible to determine, for the first time and with a reasonable degree of accuracy, the flows attained in the great floods of previous years, which occurred before or since continuous records of stage have been kept.

Long and accurate flood records are a prerequisite in any estimate of the probable magnitude and frequency of future floods. Nevertheless, in the design of hydraulic structures which are to be erected immediately, it is often necessary to make estimates of the probable frequency and magnitude of future floods from unsatisfactory records. Later stream-flow records may reveal that these estimates were sadly deficient and deviated to a startling extent from the later record of experience. Nevertheless, such estimates must be made and used as a rough guide.

This raises the question as to the methods that are commonly used for estimating possible flood flows, particularly the so-called statistical procedures. As more plentiful and accurate flood-flow data become available, it will be

Note.—The Progress Report of the Committee on Flood Protection Data was presented at the Annual Meeting, New York, N. Y., January 15, 1936, and published in February, 1936, *Proceedings*. Discussion on the report has appeared in *Proceedings*, as follows: April, 1936, by Messrs. Robert Follansbee, and LeRoy K. Sherman; August, 1936, by C. R. Pettis, M. Am. Soc. C. E.; and September, 1936, by Messrs. John C. Hoyt, and C. S. Jarvis.

[14] Asst. Hydr. Engr., Water Resources Branch, U. S. Geological Survey, Washington. D. C. Submitted with the approval of the Director, U. S. Geological Survey.

[14a] Received by the Secretary September 21, 1936.

desirable to subject these methods to very critical analysis, such as by comparing estimates obtained from various fragments of a long-time record with those obtained from the complete record. The purpose of such comparisons would be to determine the probable error in the magnitude of floods of various periodicities when estimated from records of different lengths, such as 20, 40, or 60 yr.

It will also be desirable to determine more closely the effect of widespread cyclical variations in average rainfall and run-off on flood estimates made from records which do not cover the full range of such variations. The experience of the writer, in conducting such studies and in making comparisons between the 1935 and 1936 flood flows and those obtained from frequency curves of long records (40 yr, or more), leads him to take a very cautious attitude toward all the present methods for estimating future flood flows. Not until long and accurate records which include many large floods, are available, can the present methods of flood-flow analysis be appraised properly.

MERRILL BERNARD,[15] M. AM. SOC. C. E. (by letter).[15a]—It was the writer's privilege to take part in the studies recently reported in *Water Supply Paper No. 771* entitled, "Floods in the United States—Magnitude and Frequency", in the preparation of which the Committee on Flood Protection Data rendered a signal advisory service. Concurrently with the flood study there progressed, under the leadership of W. G. Hoyt, M. Am. Soc. C. E., a study of relationships between rainfall and run-off, the results of which were published as *Water Supply Paper No. 772*, entitled "Rainfall and Run-Off in the United States." Although the latter study was under the guidance of another advisory committee, there was a profitable interchange of ideas among the two groups and their advisory committees. Not the least of the accomplishments of the study was the successful co-operation between an old-time agency of the Federal Government, advisory committees of the National societies, a co-ordinating agency of the Government (the then Mississippi Valley Committee and, later, the National Resources Committee), and the several specialists taking part in the study.

It is interesting to note that recent progress in hydrologic investigations has brought into prominence several new methods of flood-flow determination, developed upon the use of rainfall as the principal factor. Considerable time was devoted in the studies to one of these methods—the unit-hydrograph—exploring its theory and perfecting its technique. It was generally concluded, however, that further advances in this field must await the development of a successful method of dealing rationally with such factors as infiltration, evaporation, and transpiration. An appreciation of the need for such basic data is reflected in the plans of the Section of Watershed Studies, Division of Research, U. S. Soil Conservation Service, under which long-term investigations have been designed to yield quantitative data capable of areal application. All the principal phases of the hydrologic cycle have a place in these

[15] Hydr. Engr., Soil Conservation Service, Water-Shed and Hydrologic Studies, Washington, D. C.

[15a] Received by the Secretary October 5, 1936.

studies, the treatment to range from the laboratory model through experimental plots and small water-sheds, to natural drainage basins of 5 000 acres, or more.

The Committee's third recommendation, in final analysis, points to two of the several obvious inadequacies of the present system of gathering and recording rainfall data: First, the poor distribution and spacing (nationwide) of rainfall stations; and second, the inadequacy of rainfall data recorded as an average depth for 24 hr. The increasing demand for more intelligent engineering on small water-sheds, and the necessity in modern methods of flow determination to express rainfall as rate for duration periods of less than 24 hr, will, in the near future, materially restrict the use of the non-recording rain-gage. The writer believes that a recording rain-gage, simple and rugged in construction and demanding attendance not oftener than once in eight days, can be manufactured and sold for about $50. Except in the remotest regions, one man with an automobile could attend and "service" from one gage to two hundred gages spaced at optimum distances. It is believed that the additional cost of such a method of obtaining rainfall data, over that of the present system of the co-operative observer and the non-recording rain-gage, would be a good investment.

The Committee's fourth recommendation expresses, perhaps, the most urgent present-day need for hydrologic research. The Committee dispels the current erroneous belief that retardation is always the means of reducing flood peaks. The reduction, rather, is accomplished by "filling out or leveling off" the hydrograph. On water-sheds where existing conditions produce a unit hydrograph (and the resulting distribution graph) of delayed peak, such peak may be reduced by accelerating run-off from areas nearest the outlet, thereby making more efficient use of the channel system throughout the period of rising stage of the hydrograph. Further reduction in peak flow can be gained through storage and other methods of retardation, deducting from the contributions which create peak flow and adding to the contributions from the more remote areas which sustain flow throughout the falling stages of the hydrograph.

GLENN W. HOLMES,[16] ESQ. (by letter).[16a]—There seems no doubt that the Committee has made great strides toward the accomplishment of its objectives. Moreover, the recommendations submitted with its progress report have already seen fruition in library research conducted under the joint auspices of the Works Progress Administration, the U. S. Geological Survey, and the Soil Conservation Service. The results of this co-operative undertaking have been reported and are now (1936) being published by the U. S. Geological Survey as a continuation of the studies already published in *Water Supply Paper No. 771.*

It would be difficult to determine the relative values of the five recommendations made by the Committee because each is so directly involved in flood-protection work. Recommendation (3), however, is possibly the most

[16] Asst. Hydr. Engr., Soil Conservation Service, U. S. Dept. of Agriculture, Washington, D. C.

[16a] Received by the Secretary October 5, 1936.

intriguing because of the difficulties encountered in making such an inventory of cloudburst floods as proposed therein. That no systematic study of such floods has yet been undertaken may be an indication of the magnitude and difficult character of the task.

It is a conspicuous and possibly a significant fact that in most reports of cloudburst floods detailed data are lacking on the rainfall intensities and quantities that would serve to account for the often unprecedented flood crests. Precipitation data from official sources are often supplemented by information secured from unofficial observers and non-standard rainfall receivers. In the opinion of the field investigator, milk pails, chicken feed cans, hog troughs, and like containers found about rural and urban homes, provide information which, although sometimes amazing, is yet altogether necessary to account for the observed flood heights and resulting flood damage. The scientific training of the field investigator, however, often overcomes his better judgment and leads him to omit, or at least to minimize, such truly valuable data in his engineering report.

This is clearly brought out in a paper entitled, "The New York State Flood of July, 1935",[17] from which the following excerpts are quoted:

"No official records were made of the greatest and most intense rainfall that occurred."

* * * * * * * * * * * * *

"The isohyetal map on plate 22, prepared by John C. Fisher, meteorologist, United States Weather Bureau, Ithaca, shows the total rainfall recorded under the dates of July 7 and 8 at stations reporting to the United States Weather Bureau at Ithaca. Supplemental measurements of the rainfall made after the storm indicate that a much heavier precipitation than that shown on the map probably occurred over a large area."

* * * * * * * * * * * * *

"The table of 'hourly rainfall, in inches' shows the rainfall as recorded by automatic rain-gages at nine stations in or near the main storm area. Unfortunately, none of these gages was within the area of the most intense precipitation, and therefore they did not furnish information on what actually took place in that area. That at least 12 to 14 inches of rain fell in 12 to 16 hours is indicated by the amounts of rainfall measured in open receptacles after the storm."

As the writer participated in collecting and assembling some of the unofficial rainfall data mentioned, and is thoroughly convinced of their value and their fair degree of reliability, it seems appropriate to comment on certain features.

It may be noted in *Water Supply Paper No. 773-E* that the isohyetal map of the series of storms causing the flood was prepared from official records. Had the admittedly reliable supplemental data been given official recognition, a quite different rainfall map would have been made, including considerably higher values. This gives rise to two comments: (1) If such unofficial data were given fair recognition, greater efforts would be expended in collecting such information and presumably more of it

[17] "The New York State Flood of July, 1935", *Water Supply Paper No. 773-E*, U. S. Geological Survey, pp. 237 *et seq.*, 1936.

would be found; and (2) since such unofficial data are not generally recognized as acceptable, they do not become part of the published precipitation records and, therefore, do not appear among basic flood-protection data.

If adequate information cannot be secured from official sources, even in the East, where Weather Bureau Stations are comparatively close together, it seems that some alternative is necessary to insure progress along lines suggested in Recommendation (3).

The writer believes that every effort should be made to obtain the kind of unofficial data mentioned herein, and that such data should be used in making the isohyetal maps accompanying reports of cloudburst floods.

No doubt much valuable information can be obtained in this manner; but the real problems in the analysis of cloudburst floods do not wait upon choice of rainfall receivers, but rather upon the building up of a field organization which may "service" an adequate number of rain-gages wisely distributed.

When Recommendation (3) is adopted, it is probable that much higher intensities and quantities will be found than have been reported in the past, because the intensities at the centers of all storms will be recorded. These high intensities may cover areas only a few square miles in extent, or even less, but information which aids in flood-protection and run-off control for small areas is becoming increasingly important.

AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

DISCUSSIONS

WIND BRACING IN STEEL BUILDINGS

FIFTH PROGRESS REPORT OF SUB-COMMITTEE NO. 31 COMMITTEE ON STEEL OF THE STRUCTURAL DIVISION

Discussion

BY FRANCIS P. WITMER, M. AM. SOC. C. E.

FRANCIS P. WITMER,[45] M. AM. SOC. C. E. (by letter).[45a]—Under the heading "(*E*) Experimental Studies of the Behavior of Wind Bents", the conclusion was stated that, as a result of model tests at the University of Pennsylvania, vertical wind reactions for interior columns of a 4-column symmetrical bent designed for vertical loads would probably approximate zero for a rather wide range of bay ratios.

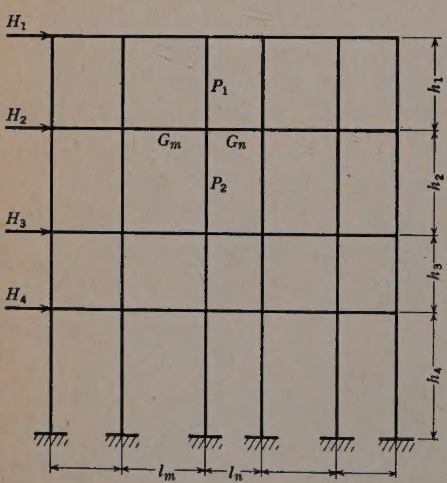

FIG. 8

The writer has since verified this conclusion by a number of additional tests, and has developed a theoretical demonstration showing that the conclusion has much more general and far-reaching application than was considered to be the case when the report was presented.

Assume a bent of the form shown in Fig. 8, with any number of columns, any number and height of stories, and any lengths of girders, and acted upon by any system of horizontal forces,

NOTE.—The Fifth Progress Report of Sub-Committee No. 31, Committee on Steel of the Structural Division on Wind-Bracing in Steel Buildings, was presented at the meeting of the Structural Division, New York, N. Y., January 16, 1936, and published in March, 1936, *Proceedings*. Discussion on this report has appeared in *Proceedings* as follows: September, 1936, by Messrs. Robins Fleming, L. E. Grinter, Chr. Nokkentved and I. Wouters.

[45] Director, Civ. Eng., Univ. of Pennsylvania, Philadelphia, Pa.

[45a] Received by the Secretary September 8, 1936.

H_1, H_2, etc. If all parts are proportional for vertical floor loads which are uniform throughout any floor (that is, if the moments of inertia of girders in any floor are proportional to the squares of their span lengths, and the moments of inertia of columns in any story are proportional to their cross-sectional areas, these, in turn, being proportional to the loads coming to them from non-continuous girders), the following conditions are true:

(1) The moments at both ends of any girder are equal;

(2) The moments in girders of any floor are proportional to their span lengths;

(3) The shears in all girders of any floor are equal to each other; and,

(4) The direct stresses and vertical reactions for all interior columns are equal to zero.

The foregoing conditions are in strict accordance with the fundamental portal theory. Consider any girder such as G_m, in Fig. 8. The moment at its right end will take the form:

$$M_{Gm} = \frac{(N_1 M_1 K_{p1} + N_2 M_2 K_{p2}) K_{Gm}}{K_{p1} + K_{p2} + K_{Gm} + K_{Gn}} \qquad (14)$$

in which, M_1 = the moment in the story above the floor in question $= H_1 h_1$; M_2 = the moment in the story below the floor in question $= (H_1 + H_2) h_2$: N_1 and N_2 are constants relating, respectively, to M_1 and M_2; and K_{p1}, K_{p2}, etc., are the stiffness ratios $\left(\frac{I}{l}\right)$ for the respective members.

From the assumptions as to column and girder moments of inertia, $K_{p1} = N_3 (l_m + l_n)$; $K_{p2} = N_4 (l_m + l_n)$; $K_{Gm} = \frac{N_5 l^2_m}{l_m} = N_5 l_m$; and, $K_{Gn} = \frac{N_5 l^2_n}{l_n} = N_5 l_n$; in which N_3 and N_4 are constants relating to all columns in the stories, h_1 and h_2, respectively, and N_5 is a constant relating to all girders in the floor in question. Therefore,

$$M_{Gm} = \frac{[N_1 M_1 N_3 (l_m + l_n) + N_2 M_2 N_4 (l_m + l_n)] N_5 l_m}{N_3 (l_m + l_n) + N_4 (l_m + l_n) + N_5 l_m + N_5 l_n} \qquad (15)$$

and,

$$M_{Gm} = \frac{(N_1 M_1 N_3 + N_2 M_2 N_4) (l_m + l_n) N_5 l_m}{(N_3 + N_4) (l_m + l_n) + N_5 (l_m + l_n)}$$

$$= \frac{(N_1 M_1 N_3 + N_2 M_2 N_4) N_5 l_m}{N_3 + N_4 + N_5} \qquad (16)$$

As the entire coefficient of l_m is a constant, which will be called N, $M_{Gm} = N l_m$, which is in accord with Condition (2).

A similar procedure at the left end of Girder G_m will produce an equal value for the moment at that end, thus verifying Condition (1) for this girder, and similar results will follow for all other girders. Condition (3) is a direct result of Conditions (1) and (2) since the shear in a member equals the sum of the end moments divided by the length of the member. Condition (4) is a necessary consequence of Condition (3). The four conditions are thus shown to be generally true.